OPEN ACCESS
International Journal of Bifurcation and Chaos, Vol. 30, No. 16 (2020) 2050240 (17 pages)

DOI: 10.1142/S0218127420502405

Breaking and Sustaining Bifurcations in S_N-Invariant Equidistant Economy

H. Aizawa*, K. Ikeda† and M. Osawa‡
Department of Civil and Environmental Engineering,
Tohoku University, Aoba, Sendai 980-8579, Japan
**hiroki.aizawa.p3@dc.tohoku.ac.jp*
†kiyohiro.ikeda.b4@tohoku.ac.jp
‡osawa.minoru.4z@kyoto-u.ac.jp

J. M. Gaspar
CEGE and Católica Porto Business School,
Universidade Católica Portuguesa,
CEF.UP, University of Porto, Portugal
jgaspar@porto.ucp.pt

Received July 19, 2019; Revised February 25, 2020

This paper elucidates the bifurcation mechanism of an equidistant economy in spatial economics. To this end, we derive the rules of secondary and further bifurcations as a major theoretical contribution of this paper. Then we combine them with pre-existing results of direct bifurcation of the symmetric group S_N [Elmhirst, 2004]. Particular attention is devoted to the existence of invariant solutions which retain their spatial distributions when the value of the bifurcation parameter changes. Invariant patterns of an equidistant economy under the replicator dynamics are obtained. The mechanism of bifurcations from these patterns is elucidated. The stability of bifurcating branches is analyzed to demonstrate that most of them are unstable immediately after bifurcation. Numerical analysis of spatial economic models confirms that almost all bifurcating branches are unstable. Direct bifurcating curves connect the curves of invariant solutions, thereby creating a mesh-like network, which appears as threads of warp and weft. The theoretical bifurcation mechanism and numerical examples of networks advanced herein might be of great assistance in the study of spatial economics.

Keywords: Bifurcation; equidistant economy; group-theoretic bifurcation theory; invariant pattern; replicator dynamics; spatial economic model; stability.

1. Introduction

Spatial economics (or economic geography) aims to explain the spatial distribution of economic activities and how distance between locations affects the economic behavior of agents. New economic geography might be regarded as a subfield of spatial economics emphasizing how economic agglomeration patterns evolve as a result of the historical increases in economic integration (i.e. decreased worldwide transportation costs). Ever since the seminal study by Krugman [1991], numerous contributions have provided insights into the connection

†Author for correspondence

between transportation costs and the spatial organization of economic activities.

Equidistant economy, in which transport costs between all pairs of places are identical, is an important spatial platform in spatial economics. Bifurcation mechanism of this economy has been investigated to observe a complicated mesh-like network of equilibrium curves [Gaspar *et al.*, 2019b]. This paper elucidates the network mechanism (1) using a group-theoretic analysis for the symmetric group S_N [Golubitsky & Stewart, 2002; Elmhirst, 2004] and (2) referring to invariant patterns for the replicator dynamics (Sec. 3 and [Ikeda *et al.*, 2018b; Ikeda *et al.*, 2019a; Ikeda *et al.*, 2019b]). It is customary in spatial economics to deal with bifurcations of steady states. Therefore, Hopf bifurcations are not considered in this paper. Hopf bifurcations with S_N-symmetry (e.g. [Golubitsky & Stewart, 1985; Diaz & Rodrigues, 2007]) and applications of bifurcation theory to dynamic problems in economics (e.g. [Dercole *et al.*, 2008; Dercole & Radi, 2020]) are to be consulted with the literature.

A *break point* and a *sustain point*, which play important roles in this discussion, were first studied in a two-place economy under replicator dynamics [Fujita *et al.*, 1999; Baldwin *et al.*, 2003]. A break point is associated with a symmetry-breaking bifurcation of two identical places, whereas a sustain point is located at the intersection of a solution curve for two identical places and a curve for two different places. Thereafter, economic agglomeration has been studied under various settings (spatial topologies), such as a line segment (e.g. [Fujita & Mori, 1997]), a racetrack (e.g. [Tabuchi & Thisse, 2011]), and a lattice (e.g. [Ikeda *et al.*, 2018a]). The equidistant case has been handled mostly with three regions [Fujita *et al.*, 1999; Castro *et al.*, 2012; Commendatore *et al.*, 2015]. Direct bifurcation from N identical places leads to a two-level hierarchy state comprising one large place and $N-1$ small places [Gaspar *et al.*, 2018]. Other works have considered an arbitrary number of equidistant places under different settings. Nevertheless, they provide only an incomplete account of the network of possible equilibria [Puga, 1999; Tabuchi *et al.*, 2005; Oyama, 2009; Zeng & Uchikawa, 2014; Gaspar *et al.*, 2019a].

In light of the discussion presented above, this paper aims to elucidate the bifurcation mechanism of an equidistant economy with N places. Bifurcation analysis of a symmetric field is a well-matured topic [Golubitsky *et al.*, 1988; Ikeda & Murota, 2019]. In fact, the mechanism of direct bifurcation of this economy is readily available through bifurcation analysis of the symmetric group S_N [Golubitsky & Stewart, 2002; Elmhirst, 2004]: all possible bifurcating solutions from the uniform state were obtained and were proven to be unstable immediately after each bifurcation. By contrast, few reports describe the secondary bifurcation under S_N-symmetry. For instance, it is analyzed as a simple example [Rodrigues, 2007]. Its stability is briefly remarked upon [Elmhirst, 2004]. A major theoretical contribution of the paper is our analysis of secondary and further bifurcations.

The existence of special spatial distributions, called *invariant patterns*, has come to be acknowledged. Steady-state solutions that satisfy the static governing equation form solution curves parameterized by the bifurcation parameter (transportation cost in spatial economics). In general, the spatial pattern changes along a solution curve. By contrast, there can be a special solution curve that has a constant spatial pattern. Such pattern is called herein an invariant pattern. Invariant patterns of a racetrack economy and of a lattice economy under the replicator dynamics, which is the most popular in economics, were found and employed to elucidate the bifurcation mechanisms of these economies [Ikeda *et al.*, 2018b; Ikeda *et al.*, 2019a; Ikeda *et al.*, 2019b]. Stability analysis of invariant patterns of an equidistant economy of spatial economic models was conducted [Gaspar *et al.*, 2018; Gaspar *et al.*, 2019a]. In this paper, invariant patterns of an equidistant economy under the replicator dynamics are found. Then the secondary (sustain) bifurcation mechanism is investigated theoretically for these invariant patterns. This theoretical stability analysis demonstrates that some of the branches are stable immediately after bifurcation, unlike the direct bifurcation.

For numerical bifurcation analysis of a symmetric system, it is customary to first obtain the uniform solution, next find direct, secondary, and further bifurcating solutions successively, and then assemble these solution curves. As described herein, because of the existence of invariant solutions, we can use the following innovative bifurcation analysis procedure [Ikeda *et al.*, 2019a]. (1) Obtain all invariant patterns and find all bifurcation points on the equilibrium curves for these patterns. (2) Obtain all direct, secondary, and further bifurcating equilibrium curves from these invariant patterns and find all bifurcation points on these curves. This procedure is put to use for numerical analysis of an equidistant economy with three, four, and eight places of two spatial economic models [Forslid & Ottaviano, 2003; Pflüger, 2004]. There are direct bifurcating curves connecting the curves of invariant solutions, thereby displaying mesh-like networks resembling threads constituting the weft of invariant patterns and the warp of noninvariant ones. Results demonstrate that almost all bifurcating equilibria are unstable.

The theoretical bifurcation mechanism and numerical examples of networks presented in this paper might be of great assistance in the study of spatial economics. In fact, this mechanism for three places [Fig. 1(a)] is applied successfully to the study of the agglomeration mechanism of a spatial economic model [Gaspar *et al.*, 2019b].

This paper is organized as explained hereinafter. A spatial economic model under the replicator dynamics is presented in Sec. 2. Invariant patterns are obtained in Sec. 3. A bifurcation mechanism of an equidistant economy is put forth in Sec. 4. The stability of bifurcating branches is studied in Sec. 5. Numerical bifurcation analyses of spatial economic models are conducted as described in Sec. 6.

2. Spatial Economic Model Under Replicator Dynamics

Economic geography models share some common components: (i) a manufacturing sector operates under imperfect competition with increasing returns to scale at the firm level; (ii) inter-regional trade of manufactured goods is costly, with transport costs; and (iii) inter-regional mobility (migration) of some production factors is allowed. Increasing returns at the firm level foster agglomeration in a place, although fiercer competition in larger markets tends to drive firms to disperse across places.

Although many economic geography models specifically examine two places, the analysis has come to be extended to multiple places, as described in the Introduction. The present study examines a setup with an arbitrary number of places that are pairwise equidistant.

Spatial economic models are presented. Their steady-state solutions under the replicator dynamics are classified. Whereas the theoretical framework of this paper is efficacious for analyzing general spatial economic models, detailed aspects of payoff functions are defined in accordance with two important models: the FO model [Forslid & Ottaviano, 2003] and the Pf model [Pflüger, 2004]. These two models serve as concrete examples of spatial economic models to be used for investigating the progress of bifurcation (Sec. 6).

2.1. *Spatial economic model*

There are N (≥ 3) places and mobile agents (workers or firms, entrepreneurs) that can choose where to locate from N places. Denote by $\mathbf{h} = \{h_i \,|\, i = 1, \ldots, N\}$ the spatial distribution of agents. It is assumed that $\sum_{i=1}^{N} h_i = 1$; accordingly, the state space is the probability simplex. The payoff (indirect utility or profit) for locating in place i is derived from a model-specific short-run general equilibrium. It is a function of goods' prices and an amount of income reflecting agents' preferences and market conditions. In equilibrium, it is given as a function v_i of the spatial distribution of mobile agents $\mathbf{h}$ and a parameter $\phi \in (0, 1)$ that represents the freeness of transport between the places. The parameter ϕ is an inverse measure of transport costs. It can be regarded as signifying the degree of economic integration between the N places. A continuous C^1 function $\mathbf{v} : \mathbb{R}^N \times (0, 1) \to \mathbb{R}^N_+$ therefore defines a general spatial economic model with N places. An *equilibrium* is defined as a spatial distribution of agents $\mathbf{h} = \mathbf{h}^*$ that satisfies the conditions of

$$\begin{cases} v^* - v_i(\mathbf{h}^*, \phi) = 0 & \text{if } h_i^* > 0, \\ v^* - v_i(\mathbf{h}^*, \phi) \geq 0 & \text{if } h_i^* = 0 \end{cases} \tag{1}$$

and $\sum_{i=1}^{N} h_i^* = 1$, where v^* denotes the equilibrium payoff level. It is noteworthy that (1) includes the case in which the payoff varies across places, because $v_i(\mathbf{h}^*, \phi) < v^*$ can possibly occur for $h_i^* = 0$. Economic backbones of the payoff function $\mathbf{v}$ for the FO and Pf models are summarized briefly below, whereas Appendix A presents related details.

Two factors, skilled and unskilled labor, are used in production along with the two sectors. There are two types of workers associated with two types of labor. The workers supply one unit of each type of labor inelastically (i.e. irrespective of the wage rate). The total endowments of skilled and unskilled workers are respectively H and L, with H normalized to unity ($H = 1$). Skilled workers are mobile across places; h_i denotes the number of skilled workers located in place i. Unskilled workers are immobile and are equally distributed across all places (i.e. the number of unskilled workers in each place is $\ell = L/N$). The two sectors are agriculture (A) and manufacturing (M). The A-sector output is homogeneous. Each unit is produced using a unit of unskilled labor under perfect competition. Trade of the A-sector good is frictionless. Choosing this good as numéraire means that its price and the wage paid to an unskilled worker are set to unity everywhere. Appendix A presents additional details. The M-sector output is a horizontally differentiated good produced using both skilled and unskilled labor under increasing returns to scale and Dixit–Stiglitz monopolistic competition [Dixit & Stiglitz, 1977]. Three major parameters exist for the models: σ represents the constant elasticity of substitution between any two manufactured goods; μ denotes the constant expenditure share on industrial varieties; and L signifies the endowment of immobile workers.

Inter-regional trade of M-sector goods incurs iceberg costs. That is, for each unit of M-sector goods transported from place i to j ($j \neq i$), only a fraction $1/\tau_{ij} < 1$ arrives. Intra-regional transport is frictionless: $\tau_{ii} = 1$ for all i. The main assumption used for these discussions is that transport costs among all pairs of places are equal, i.e. the space economy is *equidistant*.

Assumption 1. $\tau_{ij} = \tau > 1$ for all $i \neq j$.

By this assumption, the freeness of transport parameter $\phi = \tau^{1-\sigma} \in (0, 1)$ characterizes the inter-regional transport cost structure of FO and Pf models. As ϕ approaches 0 or 1, the trade cost increases or decreases, respectively. We employ ϕ as the bifurcation parameter. Derivation of the short-run general equilibrium is detailed in Appendix A. We reproduce here only the expressions of interest to us.

The payoff functions for the FO and Pf models are given as

$$\text{[FO]} \quad v_i(\mathbf{h}, \phi) = \Delta_i(\mathbf{h}, \phi)^{\frac{\mu}{\sigma-1}} w_i(\mathbf{h}, \phi), \tag{2}$$

$$\text{[Pf]} \quad v_i(\mathbf{h}, \phi) = \log \Delta_i(\mathbf{h}, \phi)^{\frac{\mu}{\sigma-1}} + w_i(\mathbf{h}, \phi), \tag{3}$$

respectively, where w_i is the wage function and

$$\Delta_i(\mathbf{h}, \phi) = \sum_{j=1}^{N} \tau_{ji}^{1-\sigma} h_j = h_i + \phi \sum_{j=1, j\neq i}^{N} h_j = h_i + \phi(1 - h_i).$$

Therein, $\tau_{ij}^{1-\sigma} = \phi$ if $i \neq j$ and $\tau_{ii}^{1-\sigma} = 1$ under Assumption 1. The wage vector $\mathbf{w} = \{w_i \,|\, 1 \leq i \leq N\}$ can be expressed explicitly as

$$\text{[FO]} \quad \mathbf{w}(\mathbf{h}, \phi) = \frac{\ell\mu}{\sigma} \left(I - \frac{\mu}{\sigma} M(\mathbf{h}, \phi) \text{diag}[\mathbf{h}] \right)^{-1} \times M(\mathbf{h}, \phi) \mathbf{1}_N, \tag{4}$$

$$\text{[Pf]} \quad \mathbf{w}(\mathbf{h}, \phi) = \frac{\mu}{\sigma} M(\mathbf{h}, \phi)(\mathbf{h} + \ell \mathbf{1}_N). \tag{5}$$

Here, $M(\mathbf{h}, \phi) = \{\tau_{ji}^{1-\sigma}/\Delta_j(\mathbf{h}, \phi) \,|\, i, j = 1, \ldots, N\}$; I represents the N-dimensional identity matrix; and $\mathbf{1}_N = \underbrace{(1, \ldots, 1)}_{N \text{ times}}$ is the N-dimensional all-one vector.

2.2. *Replicator dynamics*

It is customary in economics to replace the problem to obtain *stable* spatial equilibria in (1) with another problem to find a set of stable steady-state solutions of the replicator dynamics [Taylor & Jonker, 1978] as

$$\frac{d\mathbf{h}}{dt} = \mathbf{F}(\mathbf{h}, \phi), \tag{6}$$

where $\mathbf{F}(\mathbf{h}, \phi) = \{F_i(\mathbf{h}, \phi) \,|\, 1 \leq i \leq N\}$, and

$$F_i(\mathbf{h}, \phi) = (v_i(\mathbf{h}, \phi) - \bar{v}(\mathbf{h}, \phi))h_i. \tag{7}$$

Here, $\bar{v}(\mathbf{h}, \phi) = \sum_{i=1}^{N} h_i v_i(\mathbf{h}, \phi)$ is the average utility. Steady-state solutions (rest points) $(\mathbf{h}(\phi), \phi)$ of the replicator dynamics (6) are defined as those points which satisfy the static governing equation

$$\mathbf{F}(\mathbf{h}, \phi) = \mathbf{0}. \tag{8}$$

The law of preservation of population $\sum_{i=1}^{N} h_i = 1$ is satisfied for the steady-state solutions. Accordingly, the space of $\mathbf{h}$ is an $(N - 1)$-dimensional simplex, as explained in Remark 2.1 below. A steady-state solution is stable if every eigenvalue of the Jacobian matrix $J(\mathbf{h}, \phi) = \partial\mathbf{F}/\partial\mathbf{h}(\mathbf{h}, \phi)$ has a negative real part and is unstable if at least one eigenvalue has a positive real part. It is necessary to exclude an eigenvalue associated with an eigenvector that does not belong to the $(N-1)$-dimensional simplex. A *stable equilibrium*, which is the main target of this paper, is defined as a stable steady-state solution of (8) with non-negative population $h_i \geq 0$ ($1 \leq i \leq N$); it is known that such a solution satisfies the equilibrium condition (1) of an underlying spatial economic model [Sandholm, 2010].

Steady-state solutions are classifiable into *interior solutions* for which all places have positive population, and *corner solutions* for which some places have zero population. A corner solution can be expressed, without loss of generality, by an appropriate permutation of components of $\mathbf{h}$, as

$$\mathbf{h} = (\mathbf{h}_m^+, \mathbf{0}_n) \quad (1 \le m \le N-1; m+n=N) \tag{9}$$

with $\mathbf{h}_m^+ = \{h_i > 0, \sum_{1=1}^m h_i = 1 \,|\, 1 \le i \le m\} \in \mathbb{R}_+^m$ and the n-dimensional zero vector $\mathbf{0}_n = \underbrace{(0,\ldots,0)}_{n \text{ times}}$.

Remark 2.1. It is customary in economic geography to use $\mathbf{h} = (h_1, h_2, \ldots, h_{N-1})$ with $h_N = 1 - \sum_{i=1}^{N-1} h_i$. However, in the present formulation, all N coordinates, i.e. $\mathbf{h} = (h_1, h_2, \ldots, h_N)$, are used because the symmetry condition and the spatial population distribution can be expressed in a much more consistent manner.

3. Invariant Patterns

Invariant patterns of an equidistant economy are presented. Steady-state solutions that satisfy the static governing equation $\mathbf{F}(\mathbf{h}, \phi) = \mathbf{0}$ in (8) form solution curves $(\mathbf{h}(\phi), \phi)$ parameterized by ϕ. In general, the spatial pattern $\mathbf{h}(\phi)$ varies with ϕ along a solution curve. By contrast, by virtue of the product form (7) of the replicator dynamics, there can be a special solution curve $(\mathbf{h}(\phi), \phi) = (\bar{\mathbf{h}}, \phi)$ that has a constant spatial pattern $\mathbf{h}(\phi) = \bar{\mathbf{h}}$ along the curve. Such pattern $\bar{\mathbf{h}}$ is called herein an *invariant pattern*. The curve of an invariant pattern exists for any $\phi \in (0,1)$. On the other hand, a pattern $\mathbf{h}(\phi)$ that varies with ϕ is called a *noninvariant pattern* and might or might not be a steady state for a given ϕ.

The uniform state

$$\mathbf{h}^{\mathrm{uniform}} = \frac{1}{N}\mathbf{1}_N$$

and a series of *core–periphery* patterns (cf. Remark 3.1)

$$\mathbf{h}_m^{\mathrm{CP}} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n) \quad (1 \le m \le N-1; m+n=N) \tag{10}$$

with $\mathbf{1}_m = \underbrace{(1,\ldots,1)}_{m \text{ times}}$, play important roles in this paper as explained below.

Proposition 1. *The uniform state and the core–periphery pattern in (10) are invariant patterns for an equidistant economy.*

Proof. Appendix B.1 presents an associated proof. ■

Remark 3.1. In economic geography, the pattern $\mathbf{h} = (1, 0)$ $(N = 2)$, in which a (core) place has the whole population of 1 and another (periphery) place has no population, presents a core–periphery pattern. As described herein, this notion is extended. The pattern in (10) is called a core–periphery pattern. This pattern is a special form of the corner solution (9) with a two-level hierarchy: the identical population $1/m$ is agglomerated to m core places, whereas other n peripheral places have no population.

4. Bifurcation Mechanism

The bifurcation mechanism of secondary and further bifurcations of an equidistant economy is investigated as a novel contribution of this paper. The mechanism of the direct bifurcation of the uniform state [Golubitsky & Stewart, 2002; Elmhirst, 2004] is also included in Sec. 4.1 to make the discussion self-contained. Since it is customary in spatial economics to deal with bifurcations of steady states, this paper does not address Hopf bifurcations.

In economic geography [Fujita *et al.*, 1999], a *sustain point* is defined as the value of the transport cost parameter below which an economy with the core–periphery pattern $\mathbf{h}_1^{\mathrm{CP}} = (1, 0)$ in (10) becomes stable. In light of bifurcation theory, we can consider a sustain point as the bifurcation point at which some zero component(s) of population distribution $\mathbf{h}$ become nonzero on a bifurcating path. This kind of bifurcation is called *sustain bifurcation* herein and the associated bifurcation point is called a *sustain bifurcation point*. Similarly, we use *break bifurcation* and *break bifurcation point*.

4.1. *Direct bifurcation from a uniform state*

The mechanism of the direct bifurcation from the uniform state $\mathbf{h}^{\mathrm{uniform}} = \frac{1}{N}\mathbf{1}_N$ of an equidistant economy with N places was elucidated by the bifurcation analysis of a symmetric group S_N labeling the symmetry of this economy [Golubitsky & Stewart, 2002; Elmhirst, 2004]. This analysis is presented briefly and consistently with our formulation.

The uniform state has the Jacobian matrix of the form

$$J = A_N(a,b) = \begin{pmatrix} a & b & \cdots & b \\ b & a & b & \vdots \\ \vdots & b & \ddots & b \\ b & \cdots & b & a \end{pmatrix} \tag{11}$$

with

$$a = \frac{\partial}{\partial h_i}(v_i - \bar{v}) \quad (1 \le i \le N);$$

$$b = \frac{\partial}{\partial h_j}(v_i - \bar{v}) \quad (1 \le i, j \le N; i \ne j).$$

That is, all diagonal entries of the Jacobian matrix J are a and all off-diagonal ones are b. When $a = b$, this state encounters a direct bifurcation point with $(N-1)$-times repeated zero eigenvalues of the Jacobian matrix. At this point, several two-level hierarchy states of the form

$$\mathbf{h}_m = (\underbrace{u_1, \ldots, u_1}_{m \text{ times}}, \underbrace{u_2, \ldots, u_2}_{n \text{ times}}) = (u_1\mathbf{1}_m, u_2\mathbf{1}_n)$$
$$(1 \le m \le N-1; m+n = N; u_1 m + u_2 n = 1; u_1, u_2 > 0) \tag{12}$$

branch simultaneously in the incremental directions as

$$\delta\mathbf{h}_m = \varepsilon\left(\mathbf{1}_m, -\frac{m}{n}\mathbf{1}_n\right) \quad (1 \le m \le N-1; m+n = N; \varepsilon \in \mathbb{R}), \tag{13}$$

as explained in Proposition 2 below. Consequently, N places split into m places with equal population size and n places with another size. A branch is called *symmetric* if $\delta\mathbf{h}_m$ and $-\delta\mathbf{h}_m$ denote the same state up to a permutation of place numbers. It is designated as *asymmetric* if they do not.

Proposition 2. *At a bifurcation point of the uniform* (*equidistant*) *state, the two-level hierarchy states in* (*12*) *branch in the directions of* (*13*). *The associated branch is symmetric if* $n = m = N/2$ (N *even*); *it is asymmetric otherwise.*

Proof. Appendix B.2 and an earlier report of the literature [Elmhirst, 2004] present this proof. ■

4.2. *Bifurcation from a two-level hierarchical state*

The two-level hierarchy state $\mathbf{h}_m = (u_1\mathbf{1}_m, u_2\mathbf{1}_n)$ in (12) has break and sustain bifurcation points. At a sustain bifurcation point, where $u_2\mathbf{1}_n$ vanishes, this state exits to a corner solution expressing the core–periphery pattern in (10):

$$\mathbf{h}_m^{\mathrm{CP}} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n) \quad (1 \le m \le N-1; m+n = N).$$

For the discussion of a break bifurcation point, we refer to the Jacobian matrix of the two-level hierarchy state, which takes the form of

$$J = \begin{pmatrix} A_m(a,b) & eE_{mn} \\ fE_{nm} & A_n(c,d) \end{pmatrix}.$$

Therein, $A_m(a,b)$ and $A_n(c,d)$ are defined similarly to $A_N(a,b)$ in (11); $E_{mn} = \mathbf{1}_m{}^\top \mathbf{1}_n$ is an $m \times n$ matrix with all entries being equal to 1; and $a, b, \ldots, f$ are constants. A break bifurcation takes place when $a = b$ or $c = d$ is satisfied. We hereinafter focus on the case of $c = d$, whereas the other case of $a = b$ can be treated similarly.

At a break bifurcation point with $c = d$, a number of three-level hierarchy states of

$$\mathbf{h}_p = (u_1\mathbf{1}_m, u_2\mathbf{1}_p, u_3\mathbf{1}_{n-p})$$
$$(1 \le p \le n-1; m+n = N; u_1 m + u_2 p + u_3(n-p) = 1; u_1, u_2, u_3 > 0) \tag{14}$$

branch simultaneously in the directions of

$$\delta\mathbf{h}_p = \varepsilon(\mathbf{0}_m, (n-p)\mathbf{1}_p, -p\mathbf{1}_{n-p}) \quad (1 \le p \le n-1; m+n = N; \varepsilon \in \mathbb{R}), \tag{15}$$

as explained in Proposition 3. Consequently, n identical places split into p places with an identical population size and $n-p$ places of another size $(1 \le p \le n-1)$.

Proposition 3. *At a break bifurcation point of the two-level hierarchy state in* (*12*), *the three-level hierarchy states in* (*14*) *branch in the directions of* (*15*). *The branch is symmetric if* $p = n/2$ (n *even*) *and asymmetric otherwise.*

Proof. Appendix B.3 presents the associated proof. ■

The three-level hierarchy state in (14) can exit to the corner solution at a sustain bifurcation point; alternatively, it can undergo further bifurcations to arrive at an aggregated interior state with an s-level hierarchy of $(2 \leq s \leq N)$.

$$\mathbf{h}_{m_1,\ldots,m_s} = (u_1\mathbf{1}_{m_1}, \ldots, u_s\mathbf{1}_{m_s}) \tag{16}$$

with $\sum_{i=1}^{s} m_i = N$ and $\sum_{i=1}^{s} u_i m_i = 1$. Bifurcations can proceed until reaching a completely aggregated interior state of $h_1 > h_2 > \cdots > h_N > 0$.

4.3. *Bifurcation from core-periphery patterns*

In the discussion of bifurcation from the core–periphery pattern $\mathbf{h}^{\mathrm{CP}} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n)$ in (10), we refer to its Jacobian matrix as

$$J = \begin{pmatrix} A_m(a,b) & eE_{mn} \\ O & cI_n \end{pmatrix} \quad (1 \leq m \leq N-1) \tag{17}$$

with I_n being an $n \times n$ identity matrix and $c = v_i - \overline{v}$ $(m+1 \leq i \leq N)$ with $v_{m+1} = v_{m+2} = \cdots = v_N$. The critical point of this pattern is either a break bifurcation point for $a = b$ with singular $A_m(a,b)$ or a sustain bifurcation point for $c = 0$ with singular cI_n in (17).

Before the main discussion, we refer to the *half branch*. One must recall that the branches for the break bifurcations presented above do exist in both directions of $\delta\mathbf{h}_p$ and $-\delta\mathbf{h}_p$. By contrast, a branch exists in only one direction for a sustain bifurcation point because no negative population is allowed (Propositions 4 and 6 below); such a branch is called a *half branch*.

We start with the simplest core–periphery pattern: full agglomeration $\mathbf{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1})$, which is an invariant pattern (Proposition 1). The full agglomeration has only sustain bifurcation points, at which several bifurcating solutions of the form

$$\mathbf{h}_p = (1 - pu, u\mathbf{1}_p, \mathbf{0}_{N-p-1}) \quad \left(1 \leq p \leq N-1; 0 < u < \frac{1}{p}\right) \tag{18}$$

branch simultaneously in several directions as

$$\delta\mathbf{h}_p = \varepsilon\left(-1, \frac{1}{p}\mathbf{1}_p, \mathbf{0}_{N-p-1}\right) \quad (1 \leq p \leq N-1; \varepsilon > 0). \tag{19}$$

Proposition 4. *Full agglomeration* $\mathbf{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1})$ *does not have a limit point or a break bifurcation point but has sustain bifurcation points with the half branches in (18).*

Proof. Appendix B.4 presents an associated proof. ■

Other core–periphery patterns $(m \geq 2)$ have both break and sustain bifurcation points, which engender several bifurcating solutions as expounded in the following propositions, the proofs of which resemble that for Proposition 3.

Proposition 5. *At a break bifurcation point of the core–periphery pattern in (10), several three-level hierarchy states branch simultaneously as*

$$\mathbf{h}_p = (u_1\mathbf{1}_p, u_2\mathbf{1}_{m-p}, \mathbf{0}_n) \quad (1 \leq p \leq m-1; \; u_1 p + u_2(m-p) = 1; u_1, u_2 > 0). \tag{20}$$

The branch is symmetric if $p = m/2$ (m even) and asymmetric otherwise.

Proposition 6. *At a sustain bifurcation point of the core–periphery pattern in (10), there emerge half branches with*

$$\mathbf{h}_p = (u_1\mathbf{1}_m, u_2\mathbf{1}_p, \mathbf{0}_{n-p}) \quad (1 \leq p \leq n; u_1 m + u_2 p = 1; u_1, u_2 > 0). \tag{21}$$

The branches in (20) and (21) can encounter break and sustain bifurcation points successively to arrive at an aggregated state with an s-level hierarchy $(2 \leq s \leq N)$ as

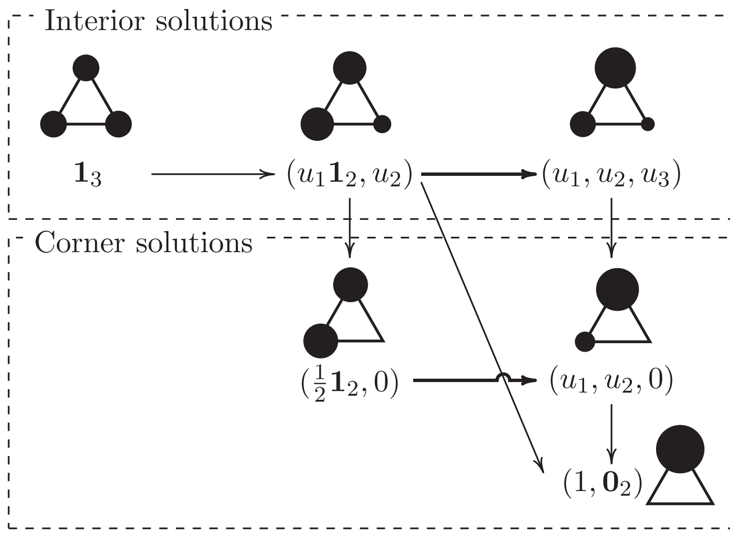
$$\mathbf{h}_{m_1,\ldots,m_s} = (u_1\mathbf{1}_{m_1}, \ldots, u_{s-1}\mathbf{1}_{m_{s-1}}, \mathbf{0}_{m_s}), \tag{22}$$

with

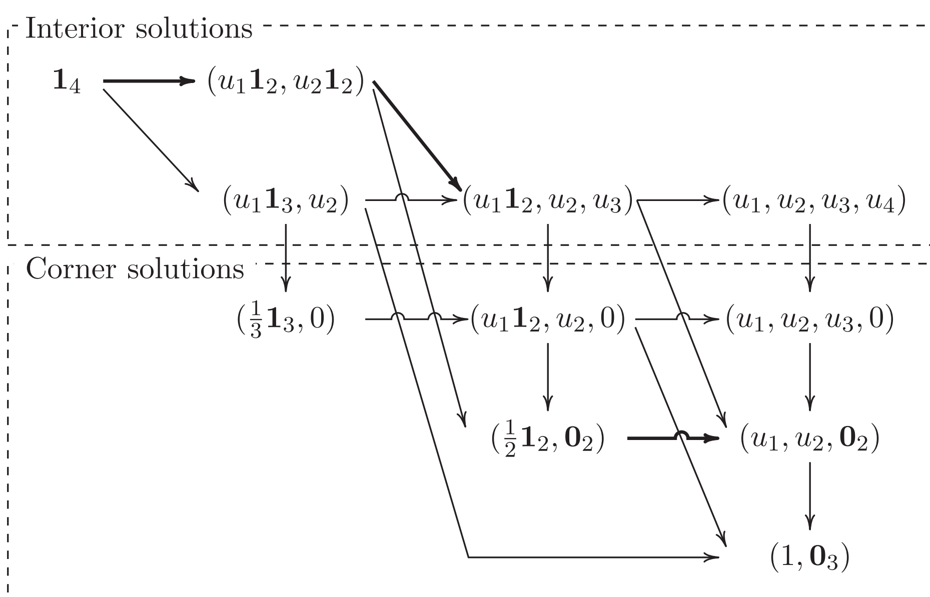
$$\sum_{i=1}^{s} m_i = N \quad \text{and} \quad \sum_{i=1}^{s-1} u_i m_i = 1.$$

4.4. *Simple examples*

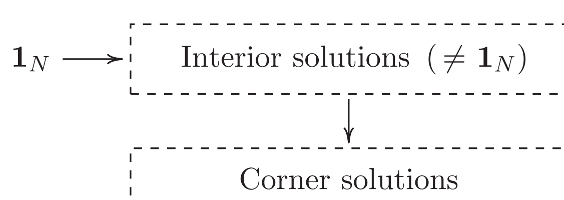
To illustrate the bifurcation mechanism in equidistant economy, Figs. 1(a) and 1(b) provide simple examples of the hierarchies of spatial patterns for the cases of $N = 3$ and $N = 4$, respectively. A symmetric branch is expressed by a thick arrow and an asymmetric one by a thin one. For each case, the subhierarchy for interior solutions at the



(a) $N = 3$



(b) $N = 4$



(c) An arbitrary N

Fig. 1. Bifurcation mechanism of an equidistant economy expressed by hierarchies of geometrical patterns. A symmetric branch is expressed by a thick arrow and an asymmetric one by a thin one.

top is connected to that for corner solutions at the bottom [see (9) for the definition of a corner solution]. There is a recurrent property: the hierarchy of $N = 3$ becomes the subhierarchy of corner solutions for $N = 4$. For an arbitrary number N of places, the subhierarchy of corner solutions is given by the hierarchy of $N - 1$ places [see Fig. 1(c)]. By virtue of this recurrent property, the hierarchy grows rapidly as N increases. In turn, the bifurcation mechanism becomes progressively complicated.

5. Asymptotic Stability of Branches

In the previous section, branches of bifurcation points for geometrical patterns of interest, such as the uniform state, the full agglomeration, and core–periphery patterns are investigated respectively in Propositions 2, 4 and 6. In this section, as a novel contribution of this paper, we investigate asymptotic stability of these branches in a close neighborhood of the associated bifurcation point. The stability of branches from the uniform state is already studied as explained below [Elmhirst, 2004].

Lemma 1. *Under the assumption that the uniform state* $\mathbf{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N$ *is stable until reaching the break bifurcation point, all branches of this state are asymptotically unstable.*

We hereinafter investigate the stability of half branches from a sustain bifurcation point of the core–periphery pattern $\mathbf{h}_m^{\text{CP}} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n)$ $(m + n = N)$ in (10). We recall its Jacobian matrix in (17) as

$$J = \begin{pmatrix} A_m(a, b) & eE_{mn} \\ O & cI_n \end{pmatrix} \quad (1 \le m \le N - 1) \tag{23}$$

and consider its sustain bifurcation point at $\phi = \phi_c$ with a singular cI_n $(c = 0)$ and a nonsingular $A_m(a, b)$ $(a \ne b)$. Define incremental variables $(\mathbf{y}, \mathbf{x}, \psi)$ from this point as

$$\mathbf{h} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n) + (\mathbf{y}, \mathbf{x}), \quad \phi = \phi_c + \psi \tag{24}$$

with $\mathbf{y} = (y_1, \ldots, y_m)$ and $\mathbf{x} = (x_1, \ldots, x_n)$. We obtain the bifurcation equation

$$\mathbf{G} = \{G_i(\mathbf{x}, \psi) \,|\, 1 \le i \le n\} = \mathbf{0} \tag{25}$$

based on the procedure: (1) Express the static governing equation $\mathbf{F}(\mathbf{h}, \phi) = \mathbf{0}$ in (8) in terms of these incremental variables $(\mathbf{y}, \mathbf{x}, \psi)$. (2) Express $\mathbf{y} = \mathbf{y}(\mathbf{x}, \psi)$ using the first m components of $\mathbf{F} = \mathbf{0}$, because $A_m(a, b)$ is nonsingular. (3) Eliminate $\mathbf{y} = \mathbf{y}(\mathbf{x}, \psi)$ from the last n components of $\mathbf{F} = \mathbf{0}$. By virtue of a factored form (7) of the replicator dynamics, $G_i(\mathbf{x}, \psi)$ takes the special form of $G_i = x_i \cdot \hat{G}_i(\mathbf{x}, \psi)$ $(1 \le i \le n)$ for some function $\hat{G}_i$. Then solutions $(\mathbf{x}, \psi)$ of $\hat{G}_1 = \cdots = \hat{G}_n = 0$ give bifurcating solutions via $\mathbf{y} = \mathbf{y}(\mathbf{x}, \psi)$ and (24).

At a sustain bifurcation point of the core–periphery pattern $\mathbf{h}_m^{\text{CP}}$, there emerge several half branches in (21), i.e.

$$\mathbf{h}_p = (u_1\mathbf{1}_m, u_2\mathbf{1}_p, \mathbf{0}_{n-p}) \qquad (1 \le p \le n; u_1 m + u_2 p = 1; u_1, u_2 > 0). \tag{26}$$

We see that these branches are associated with

$$\mathbf{x} = \varepsilon(\mathbf{1}_p, \mathbf{0}_{n-p}) \quad (\varepsilon > 0), \tag{27}$$

similarly to the proof of Proposition 3. By analysis of the bifurcation equation $\mathbf{G} = \mathbf{0}$ in (25) described in Appendix B.5, it is readily apparent that asymptotic bifurcating ψ versus ε curves exist as

$$\psi \approx -\frac{\beta + (p-1)\gamma}{\alpha}\varepsilon \quad (1 \le p \le N - 1) \tag{28}$$

with expansion coefficients α, β, and γ of the bifurcation equation. The following lemma serves in a pivotal role for the description of the stability of half branches of the core–periphery pattern $\mathbf{h}_m^{\text{CP}}$.

Lemma 2. *The eigenvalues of* $\partial\mathbf{G}/\partial\mathbf{x}$ *for the branches of the core–periphery pattern* $\mathbf{h}_m^{\text{CP}}$ *are given asymptotically as*

$$\begin{cases} \lambda_1 \approx \{\beta + (p-1)\gamma\}\varepsilon & (\textit{repeated once}), \\ \lambda_2 \approx -(\gamma - \beta)\varepsilon & (\textit{repeated } p - 1 \textit{ times}), \\ \lambda_3 \approx (\gamma - \beta)\varepsilon & (\textit{repeated } n - p \textit{ times}). \end{cases} \tag{29}$$

Proof. Appendix B.5 presents an associated proof. ■

The associated half branches are stable if all eigenvalues in (29) are negative. Note that $p = 1$ and $p = n$ are exceptional cases where λ_2 or λ_3 is absent, respectively. For these exceptional cases, stable half branches can exist as expounded below. This makes a sharp contrast with a break bifurcation point for the uniform state, for which all half branches are unstable (Lemma 1). For the description of stability, we use the following assumption, which accords with the numerical results of sustain bifurcation of the full agglomeration $\mathbf{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1})$ of spatial economic models (Sec. 6).

Assumption 2. The pre-bifurcation core–periphery pattern is stable for $\psi > 0$ ($\phi > \phi_{\mathrm{c}}$).

Proposition 7. *Under Assumption 2, the stability of half branches that emerge from* $\mathbf{h}_m^{\mathrm{CP}} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n)$ ($1 \leq m \leq N-1$) *is classifiable into three distinct cases as explained below.*

(i) *A three-level hierarchy state* $(u_1\mathbf{1}_m, u_2, \mathbf{0}_{n-1})$ *is the only stable half branch for* $\gamma < \beta < 0$. *It resides in* $\psi < 0$.
(ii) *A two-level hierarchy state* $(u_1\mathbf{1}_m, u_2\mathbf{1}_n)$ *is the only stable half branch for* $\beta < \min(\gamma, -(n-1)\gamma)$. *It resides in* $\psi < 0$.
(iii) *All half branches are unstable for* $\gamma \neq \beta$ *and* $\beta > 0$ *or* $-(n-1)\gamma < \beta < 0$.

Proof. Appendix B.6 presents an associated proof. ■

Among the plethora of half branches in (26), at most one of them is stable. Figure 2 depicts the classification of the parameter space (β, γ) into three distinct cases for the full agglomeration. For the full agglomeration state $\mathbf{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1})$, for example, the stability of half branches is readily available by setting $n = N-1$ in Proposition 7 and in Fig. 2.

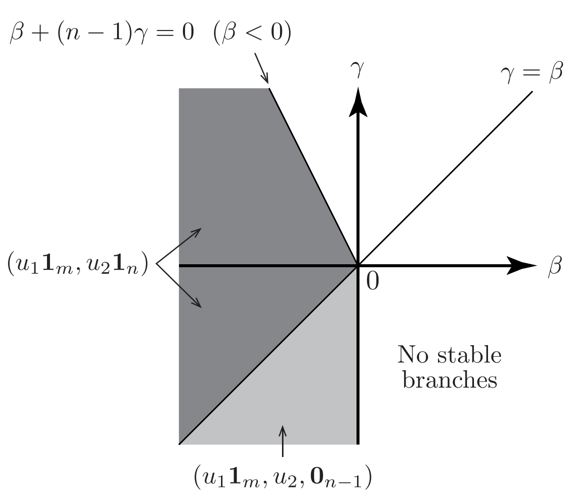


Fig. 2. Classification of stability of half branches for the core–periphery pattern $\mathbf{h}_m^{\mathrm{CP}} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n)$ in the parameter space (β, γ).

6. Numerical Bifurcation Analysis

A numerical bifurcation analysis is conducted for spatial economic models: the FO and Pf models (Sec. 2) related to theoretical results presented in Secs. 3–5. The values of the parameters in (2)–(5) are set as $(\sigma, \mu, \ell) = (6.0, 0.4, 1.0)$ for the FO model and as $(\sigma, \mu, \ell) = (4.0, 0.6, 2.0)$ for the Pf model.

The following innovative bifurcation analysis procedure is used [Ikeda *et al.*, 2019a]. (1) Obtain all invariant patterns and find all bifurcation points on the equilibrium curves for these patterns. (2) Obtain all direct, secondary, and further bifurcating equilibrium curves from these invariant patterns and find all bifurcation points on these curves. Using this procedure, one can obtain all possible bifurcating solutions connected to invariant solutions.

Figure 3 reports bifurcation diagrams for the FO and Pf models with $N = 3$, 4, and 8 equidistant places. The horizontal axis is the freeness of transport ϕ and the vertical axis is taken as $h_{\max}(\mathbf{h}) = \max_i\{h_i\}$. A series of horizontal lines expresses equilibrium curves for invariant patterns (Proposition 1):

$$\begin{cases} \text{uniform state } \mathbf{h}^{\text{uniform}} = \dfrac{1}{N}\mathbf{1}_N, \\ \text{core–periphery pattern} \\ \qquad \mathbf{h}_m^{\mathrm{CP}} = \dfrac{1}{m}(\mathbf{1}_m, \mathbf{0}_n) \quad (2 \leq m \leq N-1), \\ \text{full agglomeration } \mathbf{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1}). \end{cases}$$

The solid (broken) curves correspond to stable (unstable) steady-state solutions of the governing equation (8). The white circles (○) in the figures denote break bifurcation points, whereas the black disks (●) denote sustain bifurcation points. The double circle (◎) in Fig. 3(b) represents a limit point of ϕ. The stability analysis of the direct bifurcating curves was conducted to confirm that there are no bifurcation points on these curves. Accordingly, there are no secondary bifurcations from these curves. Figure 3 therefore encompasses all possible equilibrium curves connected to the curves of invariant patterns.

The break bifurcation point A (○) is located at the right end of the solid horizontal line with $h_{\max} = \frac{1}{N}$ of the stable uniform state $\mathbf{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N$. Several two-level hierarchy states branching

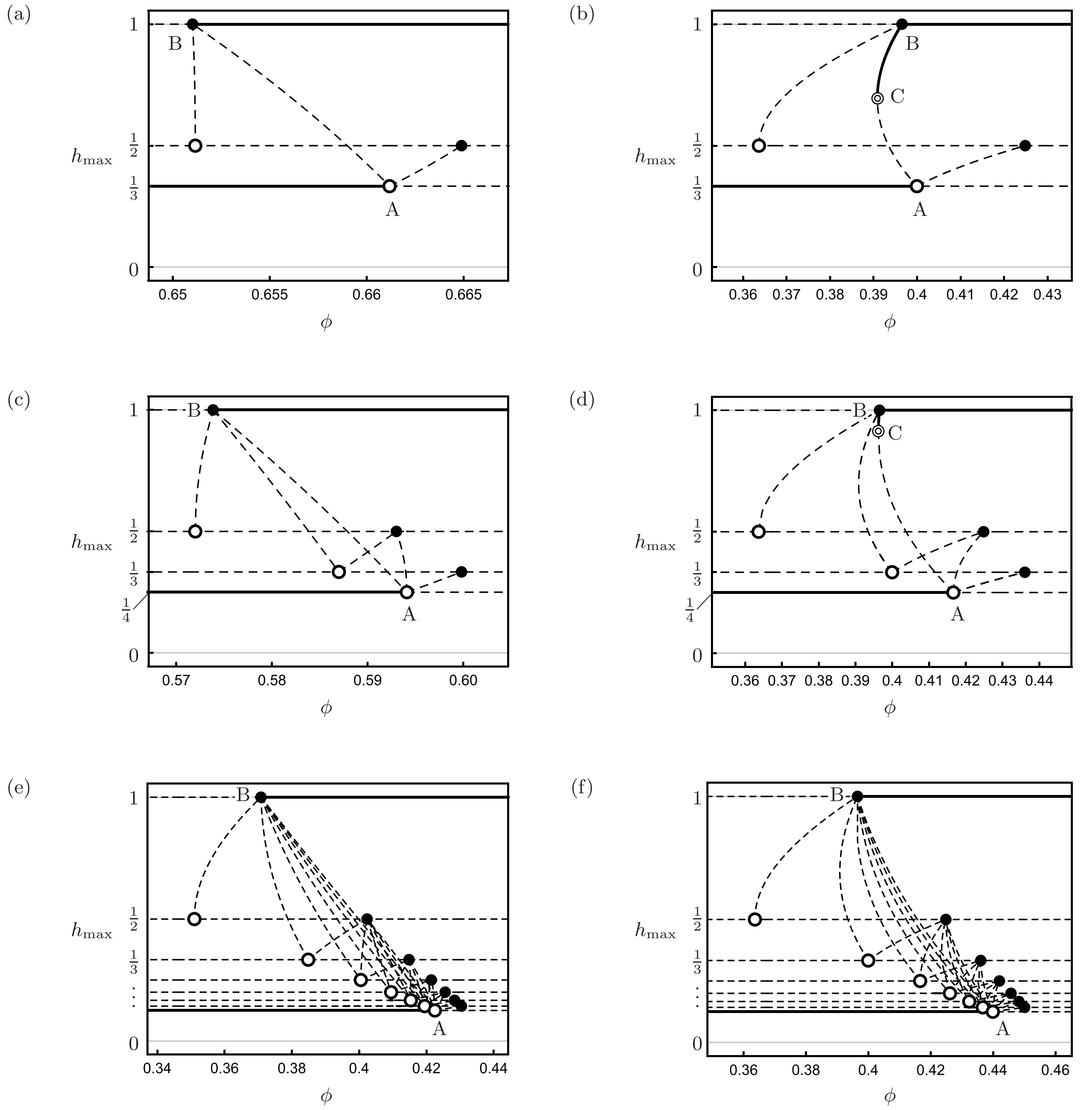


Fig. 3. Bifurcation diagrams for $N = 3$, 4, and 8 for the models by Forslid and Ottaviano [2003] and Pflüger [2004]: solid line, stable steady state; broken line, unstable steady state; ○, break bifurcation point; •, sustain bifurcation point; ◎, limit point of ϕ. (a) Forslid and Ottaviano [2003], $N = 3$, (b) Pflüger [2004], $N = 3$, (c) Forslid and Ottaviano [2003], $N = 4$, (d) Pflüger [2004], $N = 4$, (e) Forslid and Ottaviano [2003], $N = 8$ and (f) Pflüger [2004], $N = 8$.

from this point (Proposition 2) exist as

$$\mathbf{h}_m = (u_1\mathbf{1}_m, u_2\mathbf{1}_n) \quad (1 \le m \le N-1; \quad m+n=N; u_1 m + u_2 n = 1).$$

These states connect the break bifurcation point A (○) of the uniform state $\mathbf{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N$ with $N-1$ sustain bifurcation points (•) of core–periphery patterns in (10):

$$\mathbf{h}_m^{\text{CP}} = \frac{1}{m}(\mathbf{1}_m, \mathbf{0}_n) \quad (1 \le m \le N-1; m+n=N).$$

From each of break bifurcation points (○) of the two-level hierarchy states with $h_{\max} = \frac{1}{m}$ $(1 \le m \le$

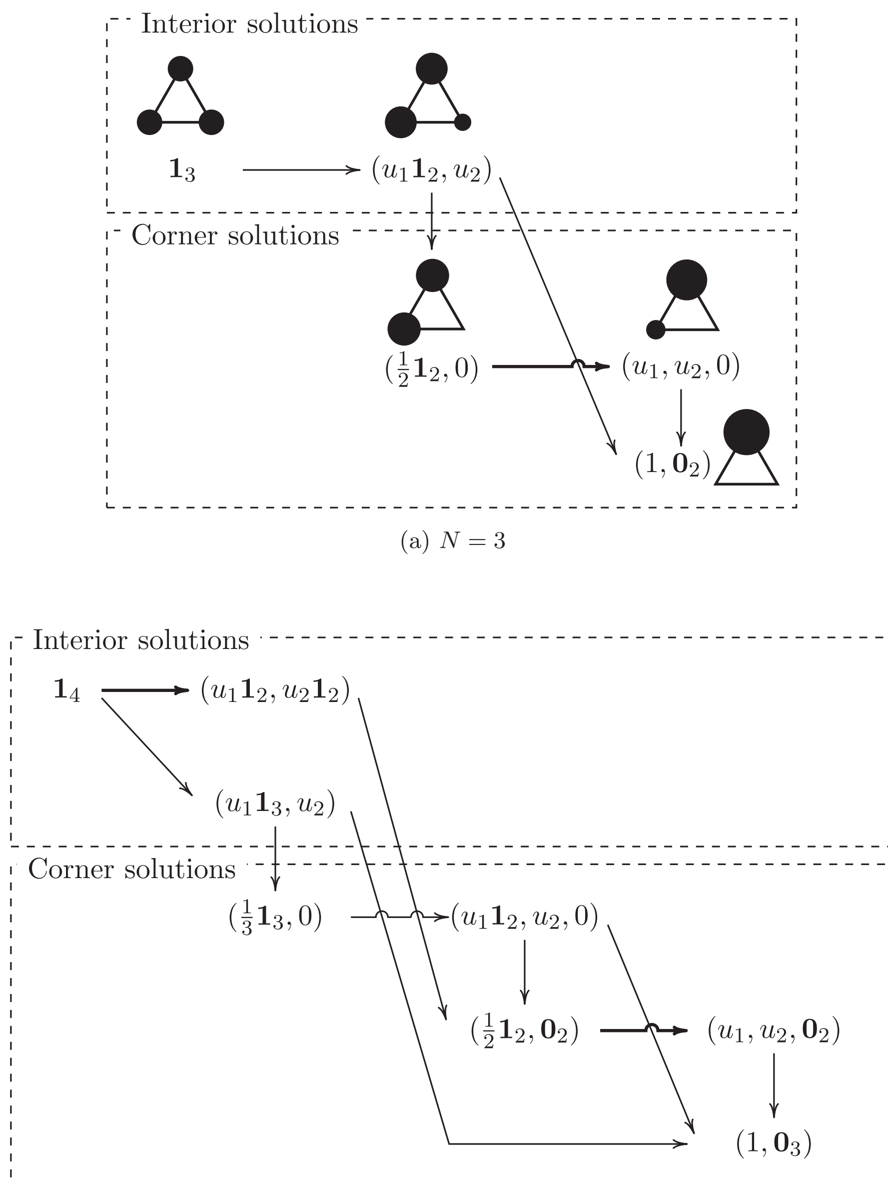
$N-1$), several branches with three-level hierarchy states [(20) in Proposition 5] emerge as

$$\mathbf{h}_p = (u_1\mathbf{1}_p, u_2\mathbf{1}_{m-p}, \mathbf{0}_n)$$
$$(1 \le p \le m-1; u_1 p + u_2(m-p) = 1).$$

Each of these states connects a break bifurcation point (∘) with a sustain bifurcation point (•). Such pairs of break and sustain bifurcation points were encountered recurrently until reaching the full agglomeration $\mathbf{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1})$ at the sustain bifurcation point B which resides at the left end of the solid horizontal line for stable full agglomeration.

Direct bifurcating curves connecting invariant solutions thereby display mesh-like networks, which resemble threads constituting the weft of invariant patterns and warp of noninvariant ones. These networks are much clearer and more systematic than that for a hexagonal lattice [Ikeda *et al.*, 2019a], possibly by virtue of a large symmetry of the symmetric group S_N.

Among invariant patterns, only the uniform state and full agglomeration have some stable equilibria. All the branches from the uniform state $\mathbf{h}^{\mathrm{uniform}} = \frac{1}{N}\mathbf{1}_N$ are unstable immediately after



(a) $N = 3$

(b) $N = 4$

Fig. 4. Hierarchies of geometrical patterns in numerical analyses for $N = 3$ and 4. A symmetric branch is expressed by a thick arrow and an asymmetric one is expressed by a thin arrow.

bifurcation (Lemma 1). Curve BC in Fig. 3 for a noninvariant branch of the Pf model is stable as predicted in Proposition 7(ii); other cases include no stable, noninvariant curve.

Figure 4 depicts the hierarchy of geometrical patterns for the present numerical analyses. The hierarchy diagram of the $N = 3$ case is a "subset" of the $N = 4$ case. These hierarchies correspond to the subsets of theoretical hierarchies summarized by Fig. 1. As demonstrated by Figs. 3(e) and 3(f) for $N = 8$, the hierarchy grows rapidly as N increases.

7. Conclusions

A thorough study was conducted on the bifurcation mechanism and stability of an equidistant economy. As a novel contribution, we elucidated the bifurcation mechanisms of secondary and further bifurcations. The bifurcation mechanism of the direct bifurcation of the uniform state [Golubitsky & Stewart, 2002; Elmhirst, 2004] is also included to make the discussion self-contained. The equilibrium curves of this economy have complicated mesh-like networks comprising intersecting equilibrium curves of invariant and noninvariant patterns, similar to threads of warp and weft. The bifurcation mechanisms advanced in this study were of great assistance in obtaining these complicated curves. The theoretical bifurcation mechanisms and numerical examples of networks advanced in this paper are expected to be of great assistance in the study of multiple places in spatial economics.

Acknowledgments

We thank two anonymous reviewers and editors for useful comments and suggestions. We gratefully acknowledge Grants-in-Aid received from JSPS 18K04380/18K18874/19K15108 and financial support from Fundação para a Ciência e Tecnologia (through projects PTDC/EGE-ECO/30080/2017, CEECIND/02741/2017, UIDB/04105/2020 and UIDB/00731/202).

References

Baldwin, R., Forslid, R., Martin, P., Ottaviano, G. & Robert-Nicoud, F. [2003] *Economic Geography and Public Policy* (Princeton University Press, Princeton).

Castro, S. B. S. D., Correia-da-Silva, J. & Mossay, P. [2012] "The core-periphery model with three regions and more," *Pap. Reg. Sci.* **91**, 401–418.

Commendatore, P., Kubin, I. & Sushko, I. [2015] "Typical bifurcation scenario in a three region identical new economic geography model," *Math. Comput. Simulat.* **108**, 63–80.

Dercole, F., Dieckmann, U., Obersteiner, M. & Rinaldi, S. [2008] "Adaptive dynamics and technological change," *Technovation* **28**, 335–348.

Dercole, F. & Radi, D. [2020] "Does the 'uptick rule' stabilize the stock market? Insights from adaptive rational equilibrium dynamics," *Chaos Solit. Fract.* **130**, 109426.

Diaz, A. P. S. & Rodrigues, A. [2007] "Hopf bifurcation with S_N-symmetry," *Nonlinearity* **22**, 627–666.

Dixit, A. & Stiglitz, J. [1977] "Monopolistic competition and optimum product diversity," *Am. Econ. Rev.* **67**, 297–308.

Elmhirst, T. [2004] "S_N-equivariant symmetry-breaking bifurcations," *Int. J. Bifurcation and Chaos* **14**, 1017–1036.

Forslid, R. & Ottaviano, G. I. P. [2003] "An analytically solvable core-periphery model," *J. Econ. Geogr.* **3**, 229–340.

Fujita, M. & Mori, T. [1997] "Structural stability and the evolution of urban systems," *Reg. Sci. Urban Econ.* **42**, 399–442.

Fujita, M., Krugman, P. & Venables, A. J. [1999] *The Spatial Economy* (MIT Press, Cambridge).

Gaspar, J. M., Castro, S. B. & Correia-da-Silva, J. [2018] "Agglomeration patterns in a multi-regional economy without income effects," *Econ. Theory* **66**, 863–899.

Gaspar, J. M., Castro, S. B. S. D. & Correia-da-Silva, J. [2019a] "The Footloose Entrepreneur model with a finite number of equidistant regions," *Int. J. Econ. Th.*, forthcoming.

Gaspar, J. M., Ikeda, K. & Onda, M. [2019b] "Global bifurcation mechanism and local stability of identical and equidistant regions: Application to three regions and more," *Reg. Sci. Urban Econ.*, forthcoming, https://doi.org/10.1016/j.regsciurbeco.2020.103597.

Golubitsky, M. & Stewart, I. [1985] "Hopf bifurcation in the presence of symmetry," *Arch. Ration. Mech. Anal.* **87**, 107–165.

Golubitsky, M., Stewart, I. & Schaeffer, D. G. [1988] *Singularities and Groups in Bifurcation Theory*, Vol. 2 (Springer, NY).

Golubitsky, M. & Stewart, I. [2002] *The Symmetry Perspective* (Birkhäuser, Basel).

Ikeda, K., Aizawa, H., Kogure, Y. & Takayama, Y. [2018a] "Stability of bifurcating patterns of spatial economy models on a hexagonal lattice," *Int. J. Bifurcation and Chaos* **28**, 1850138-1–30.

Ikeda, K., Onda, M. & Takayama, Y. [2018b] "Spatial period doubling, invariant pattern, and break point in economic agglomeration in two dimensions," *J. Econ. Dyn. Contr.* **92**, 129–152.

Ikeda, K. & Murota, K. [2019] *Imperfect Bifurcation in Structures and Materials*, 3rd edition (Springer, NY).

Ikeda, K., Kogure, Y., Aizawa, H. & Takayama, Y. [2019a] "Invariant patterns for replicator dynamics on a hexagonal lattice," *Int. J. Bifurcation and Chaos* **29**, 1930014-1–17.

Ikeda, K., Onda, M. & Takayama, Y. [2019b] "Bifurcation theory of a racetrack economy in a spatial economy model," *Netw. Spat. Econ.* **9**, 57–82.

Krugman, P. [1991] "Increasing returns and economic geography," *J. Polit. Econ.* **99**, 483–499.

Oyama, D. [2009] "Agglomeration under forward-looking expectations: Potentials and global stability," *Reg. Sci. Urban Econ.* **39**, 696–713.

Pflüger, M. [2004] "A simple, analytically solvable, Chamberlinian agglomeration model," *Reg. Sci. Urban Econ.* **34**, 565–573.

Puga, D. [1999] "The rise and fall of regional inequalities," *Eur. Econ. Rev.* **43**, 303–334.

Rodrigues, A. [2007] "Bifurcation of dynamical systems with symmetry," PhD thesis, Faculdade de Ciência da Universifade do Porto.

Sandholm, W. H. [2010] *Population Games and Evolutionary Dynamics* (MIT Press, Cambridge).

Tabuchi, T., Thisse, J.-F. & Zeng, D.-Z. [2005] "On the number and size of cities," *J. Econ. Geogr.* **5**, 423–448.

Tabuchi, T. & Thisse, J. F. [2011] "A new economic geography model of central places," *J. Urban Econ.* **69**, 240–252.

Taylor, P. D. & Jonker, L. B. [1978] "Evolutionary stable strategies and game dynamics," *Math. Biosci.* **40**, 145–156.

Zeng, D.-Z. & Uchikawa, T. [2014] "Ubiquitous inequality: The home market effect in a multicountry space," *J. Math. Econ.* **50**, 225–233.

Appendices

Appendix A

Core–Periphery Models

Two core–periphery models, the FO model and the Pf model, are introduced. In these models, each worker chooses a consumption level that maximizes the worker's own utility, given the spatial distribution $\mathbf{h}$ of workers. This process, the so-called "short-run equilibrium," determines the payoff $\mathbf{v} = (v_i \mid i = 1, 2, \ldots, N)$ of workers as a function of $\mathbf{h}$.

Individuals share the same utility function U over the M-sector and A-sector goods. The utility function in place i is

$$\text{[FO model]}\quad U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = (C_i^{\mathrm{M}})^{\mu}(C_i^{\mathrm{A}})^{1-\mu} \quad (0 < \mu < 1), \qquad \text{(A.1a)}$$

$$\text{[Pf model]}\quad U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + C_i^{\mathrm{A}} \quad (\mu > 0), \qquad \text{(A.1b)}$$

where μ is a constant parameter, C_i^{A} represents the consumption of the A-sector product in place i, and C_i^{M} is the manufacturing aggregate in place i, which is defined as

$$C_i^{\mathrm{M}} \equiv \left(\sum_j \int_0^{n_j} q_{ji}(\xi)^{(\sigma-1)/\sigma} d\xi \right)^{\sigma/(\sigma-1)}.$$

Therein, $q_{ji}(\xi)$ stands for the consumption in place i of a variety $\xi \in [0, n_j]$ produced in place j, n_j denotes the continuum range of varieties produced in place j, often called the number of available varieties, and $\sigma > 1$ expresses the constant elasticity of substitution between any two varieties. The budget constraint is given as

$$p_i^{\mathrm{A}} C_i^{\mathrm{A}} + \sum_j \int_0^{n_j} p_{ji}(\xi) q_{ji}(\xi) d\xi = Y_i, \qquad \text{(A.2)}$$

where p_i^{A} represents the price of the A-sector good in place i, $p_{ji}(\xi)$ signifies the price of a variety ξ in place i produced in place j and Y_i stands for the income of an individual in place i. The incomes (wages) of skilled workers and unskilled workers are denoted, respectively, by w_i and w_i^{L}.

An individual in place i maximizes (A.1) subject to (A.2). This maximization yields the following demand functions:

$$\text{[FO model]}\quad C_i^{\mathrm{A}} = (1-\mu)\frac{Y_i}{p_i^{\mathrm{A}}}, \quad C_i^{\mathrm{M}} = \mu \frac{Y_i}{\rho_i},$$

$$q_{ji}(\xi) = \mu \frac{\rho_i^{\sigma-1} Y_i}{p_{ji}(\xi)^{\sigma}}, \qquad \text{(A.3a)}$$

$$\text{[Pf model]}\quad C_i^{\mathrm{A}} = \frac{Y_i}{p_i^{\mathrm{A}}} - \mu, \quad C_i^{\mathrm{M}} = \mu \frac{p_i^{\mathrm{A}}}{\rho_i},$$

$$q_{ji}(\xi) = \mu \frac{p_i^{\mathrm{A}} \rho_i^{\sigma-1}}{p_{ji}(\xi)^{\sigma}}. \qquad \text{(A.3b)}$$

Therein,

$$\rho_i = \left(\sum_j \int_0^{n_j} p_{ji}(\xi)^{1-\sigma} d\xi \right)^{1/(1-\sigma)} \qquad \text{(A.4)}$$

denotes the price index of the differentiated product in place i. Because the total income and population in place i are $w_i h_i + w_i^{\mathrm{L}} \ell$ and $h_i + \ell$, respectively, we have total demand $Q_{ji}(\xi)$ in place i for a variety ξ produced in place j as

$$\text{[FO model]}\quad Q_{ji}(\xi) = \mu \frac{\rho_i^{\sigma-1}}{p_{ji}(\xi)^\sigma}(w_i h_i + w_i^{\mathrm{L}}\ell), \tag{A.5a}$$

$$\text{[Pf model]}\quad Q_{ji}(\xi) = \mu \frac{p_i^{\mathrm{A}}\rho_i^{\sigma-1}}{p_{ji}(\xi)^\sigma}(h_i + \ell). \tag{A.5b}$$

The A sector is perfectly competitive. It produces homogeneous goods under constant-returns-to-scale technology, which requires one unit of unskilled labor to produce one unit of output. For simplicity, we assume that the A-sector goods are transported between places without transportation cost and assume that they are chosen as numéraire. These assumptions mean that, in equilibrium, the wage of an unskilled worker w_i^{L} is equal to the price of A-sector goods in all places (i.e. $p_i^{\mathrm{A}} = w_i^{\mathrm{L}} = 1$ for each $i = 1, \ldots, N$).

The M-sector output is produced under increasing returns to scale technology and under Dixit–Stiglitz monopolistic competition. A firm incurs a fixed input requirement of α units of skilled labor and a marginal input requirement of β units of unskilled labor. That is, linear technology in terms of unskilled labor is assumed in the profit function. Given the fixed input requirement α, skilled labor market clearing implies $n_i = h_i/\alpha$ in equilibrium. An M-sector firm located in place i chooses $(p_{ij}(\xi) \,|\, j = 1, \ldots, N)$, which maximizes its profit

$$\Pi_i(\xi) = \sum_j p_{ij}(\xi) Q_{ij}(\xi) - (\alpha w_i + \beta x_i(\xi)),$$

where $x_i(\xi)$ is the total supply.

Recall that the transportation costs for M-sector goods are assumed to take the iceberg form. That is, for each unit of M-sector goods transported from place i to place j ($\neq i$), only a fraction $1/\tau_{ij} < 1$ arrives ($\tau_{ii} = 1$). Consequently, the total supply $x_i(\xi)$ is

$$x_i(\xi) = \sum_j \tau_{ij} Q_{ij}(\xi). \tag{A.6}$$

Because a continuum of firms exists, each firm is negligible in the sense that its action has no effect on the market (i.e. the price indices). Therefore, the first-order condition for profit maximization yields

$$p_{ij}(\xi) = \frac{\sigma\beta}{\sigma - 1}\tau_{ij}. \tag{A.7}$$

This expression implies that the price of the M-sector products is independent of variety ξ. Consequently, $Q_{ij}(\xi)$ and $x_i(\xi)$ are independent of ξ. Therefore, the argument ξ is suppressed in the sequel. Substituting (A.7) into (A.4), we have the price index

$$\rho_i = \frac{\sigma\beta}{\sigma - 1}\left(\frac{1}{\alpha}\sum_j h_j \phi_{ji}\right)^{1/(1-\sigma)}, \tag{A.8}$$

where $\phi_{ji} = \tau_{ji}^{1-\sigma}$ is a spatial discounting factor between places j and i from (A.5) and (A.8), ϕ_{ji} is obtained as $(p_{ji}Q_{ji})/(p_{ii}Q_{ii})$, which means that ϕ_{ji} is the ratio of total expenditure in place i for each M-sector product produced in place j to the expenditure for a domestic product. Under our assumptions, $\phi_{ii} = 1$ and $\phi_{ij} = \phi = \tau^{1-\sigma}$ for all $i \neq j$.

In the short run, skilled workers are immobile between places, i.e. their spatial distribution ($\mathbf{h} = (h_i) \in \mathbb{R}^N$) is assumed to be given. The market equilibrium conditions consist of the M-sector goods market clearing condition and the zero-profit condition because of the free entry and exit of firms. The former condition can be expressed as (A.6). The latter condition requires that the operating profit of a firm be absorbed entirely by the wage bill of its skilled workers as

$$w_i(\mathbf{h}, \phi) = \frac{1}{\alpha}\left(\sum_j p_{ij} Q_{ij}(\mathbf{h}, \phi) - \beta x_i(\mathbf{h}, \phi)\right). \tag{A.9}$$

Substituting (A.5)–(A.8) into (A.9), we have the market equilibrium wage of

$$\text{[FO model]}\quad w_i(\mathbf{h}, \phi) = \frac{\mu}{\sigma}\sum_j \frac{\phi_{ij}}{\Delta_j(\mathbf{h}, \phi)} \times (w_j(\mathbf{h}, \phi) h_j + 1), \tag{A.10a}$$

$$\text{[Pf model]}\quad w_i(\mathbf{h}, \phi) = \frac{\mu}{\sigma}\sum_j \frac{\phi_{ij}}{\Delta_j(\mathbf{h}, \phi)}(h_j + 1), \tag{A.10b}$$

where $\Delta_j(\mathbf{h}, \phi) \equiv \sum_k \phi_{kj} h_k$ denotes the market size of the M-sector in place j. Consequently, $\phi_{ij}/\Delta_j(\mathbf{h}, \phi)$ defines the market share in place j of each M-sector product produced in place i.

The indirect utility $v_i(\mathbf{h}, \phi)$ in the main text, given the spatial distribution of the skilled workers, is obtained by substituting (A.3), (A.8), and (A.10) into (A.1).

Appendix B

Proof of Propositions and a Lemma

B.1. *Proof of Proposition 1*

For the uniform state, we have $(v_i - \bar{v})h_i = 0$ $(1 \leq i \leq N)$ because $v_1 = \cdots = v_N = \bar{v}$. Accordingly, this state always satisfies the static governing equation $\mathbf{F}(\mathbf{h}^*, \phi) = \mathbf{0}$ in (8). For the core–periphery pattern $\mathbf{h}_m^{\text{CP}} = (\frac{1}{m}\mathbf{1}, \mathbf{0}_n)$, we have $(v_i - \bar{v})h_i = 0$ $(m+1 \leq i \leq N)$ for zero components $\mathbf{0}_n$ of $\mathbf{h}_m^{\text{CP}}$. For the components $\frac{1}{m}\mathbf{1}$, we have $v_1 = v_2 = \cdots = v_m$ and

$$\begin{aligned}\bar{v} &= \sum_{i=1}^{m} h_i v_i + \sum_{i=m+1}^{N} h_i v_i \\ &= \left(\sum_{i=1}^{m} \frac{1}{m}\right) v_1 + \sum_{i=m+1}^{N} 0 \times v_i = v_1.\end{aligned}$$

Then $(v_i - \bar{v})h_i = (v_1 - \bar{v})h_1 = 0$ $(1 \leq i \leq m)$. Consequently, the core–periphery pattern is a steady-state solution for any ϕ.

B.2. *Proof of Proposition 2*

Consider the uniform state $\mathbf{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N$, which is invariant to the symmetric group S_N, and a state with the symmetry of an axial subgroup $\text{S}_m \times \text{S}_n$ $(m + n = N)$. Denote by

$$\delta\mathbf{h} = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$$

an incremental variable vector for this state with $\text{S}_m \times \text{S}_n$ symmetry. By S_m and S_n symmetries, we have

$$\alpha_1 = \cdots = \alpha_m = \alpha, \quad \beta_1 = \cdots = \beta_n = \beta$$

for some variables α and β. By virtue of the orthogonality between subspaces for S_N and $\text{S}_m \times \text{S}_n$ $(m + n = N)$, we have

$$\begin{aligned}\mathbf{h}^{\text{uniform}}\delta\mathbf{h}^\top &= \frac{1}{N}\mathbf{1}_N(\alpha\mathbf{1}_m, \beta\mathbf{1}_n)^\top \\ &= \frac{1}{N}(\alpha m + \beta n) = 0.\end{aligned}$$

Hence $\beta = -\frac{m}{n}\alpha$ and

$$\delta\mathbf{h} = \alpha\left(\mathbf{1}_m, -\frac{m}{n}\mathbf{1}_n\right) \tag{B.1}$$

spans a one-dimensional space. Then by the equivariant branching lemma [Golubitsky *et al.*, 1988; Ikeda & Murota, 2019], a bifurcating solution exists in the direction (B.1), i.e. (13). A bifurcating solution takes the form:

$$\begin{aligned}\mathbf{h} &= \frac{1}{N}\mathbf{1}_N + \delta\mathbf{h} \\ &= \left(\left(\frac{1}{N} + \alpha\right)\mathbf{1}_m, \left(\frac{1}{N} - \alpha\frac{m}{n}\right)\mathbf{1}_n\right) \\ &= (u_1\mathbf{1}_m, u_2\mathbf{1}_n)\end{aligned}$$

with $u_1 = \frac{1}{N} + \alpha$ and $u_2 = \frac{1}{N} - \alpha\frac{m}{n}$, thereby showing (12).

The branch for (B.1) is symmetric if $m = n$ because $\delta\mathbf{h} = \alpha(\mathbf{1}_m, -\mathbf{1}_m)$ and $-\delta\mathbf{h} = \alpha(-\mathbf{1}_m, \mathbf{1}_m)$ $(N = 2m)$ are identical up to the permutation of components. It is asymmetric otherwise because the number of positive components is different for $\delta\mathbf{h}$ and $-\delta\mathbf{h}$.

B.3. *Proof of Proposition 3*

Consider the uniform state $\mathbf{h}^{\text{uniform}} = \frac{1}{N}\mathbf{1}_N$ with the symmetry of S_N, a two-level hierarchy state $\mathbf{h}^* = (u_1\mathbf{1}_m, u_2\mathbf{1}_n)$ with the symmetry of $\text{S}_m \times \text{S}_n$, and a three-level hierarchy state with the symmetry of $\text{S}_m \times \text{S}_{n_1} \times \text{S}_{n_2}$ $(m + n = N; n_1 + n_2 = n)$. One can denote by

$$\delta\mathbf{h} = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_{n_1}, \gamma_1, \ldots, \gamma_{n_2}) \tag{B.2}$$

an incremental variable vector for this state with $\text{S}_m \times \text{S}_{n_1} \times \text{S}_{n_2}$ symmetry. By S_m, S_{n_1}, and S_{n_2} symmetries, we have

$$\begin{aligned}\alpha_1 = \cdots = \alpha_m = \alpha, \\ \beta_1 = \cdots = \beta_{n_1} = \beta, \\ \gamma_1 = \cdots = \gamma_{n_2} = \gamma,\end{aligned}$$

for some variables α, β, and γ. By virtue of the orthogonality between subspaces for S_N, $\text{S}_m \times \text{S}_n$, and $\text{S}_m \times \text{S}_{n_1} \times \text{S}_{n_2}$, we have

$$\begin{aligned}\mathbf{h}^{\text{uniform}}\delta\mathbf{h}^\top &= \frac{1}{N}\mathbf{1}_N(\alpha\mathbf{1}_m, \beta\mathbf{1}_{n_1}, \gamma\mathbf{1}_{n_2})^\top \\ &= \frac{1}{N}(\alpha m + \beta n_1 + \gamma n_2) \\ &= 0, \\ \mathbf{h}^*\delta\mathbf{h}^\top &= (u_1\mathbf{1}_m, u_2\mathbf{1}_n)(\alpha\mathbf{1}_m, \beta\mathbf{1}_{n_1}, \gamma\mathbf{1}_{n_2})^\top \\ &= u_1\alpha m + u_2(\beta n_1 + \gamma n_2) \\ &= 0.\end{aligned}$$

Thus we have $\alpha = 0$ and $\gamma = -\frac{n_1}{n_2}\beta$. Then (B.2) becomes

$$\delta\mathbf{h} = \beta\left(\mathbf{0}_m, \mathbf{1}_{n_1}, -\frac{n_1}{n_2}\mathbf{1}_{n_2}\right).$$

Because this is spanned by a one-dimensional space, by the equivariant branching lemma, there exists a bifurcating solution in this direction $\delta\mathbf{h}$, which leads to (15) by setting $\beta = \varepsilon(n-p)$, $n_1 = p$, and $n_2 = n-p$. Equation (14) and symmetry/asymmetry of the branch can be proved similarly to the proof presented for Proposition 2.

B.4. *Proof of Proposition 4*

The eigenvalue of the Jacobian matrix of the full agglomeration $\mathbf{h}^{\mathrm{FA}} = (1, \mathbf{0}_{N-1})$ for a limit point or a break bifurcation point is given by

$$A_1 = a = \frac{\partial(v_1 - \bar{v})}{\partial h_1} = -v_1 < 0.$$

Because v_1 is assumed to be positive (Sec. 2.1), this eigenvalue is always nonzero (negative). Accordingly, a limit point or a break bifurcation point does not exist. The proof for the half branches of the sustain bifurcation point is similar to that presented for Proposition 3.

B.5. *Proof of Lemma 2*

The asymptotic forms of G_i in (25) and its Jacobian matrix $\partial G_i/\partial x_j$ are given as explained below. By virtue of a factored form (7) of the replicator dynamics, $G_i(\mathbf{x}, \psi)$ takes a special form of $G_i = x_i \cdot \hat{G}_i(\mathbf{x}, \psi)$ $(1 \le i \le n)$. We can expand $\hat{G}_i$ into a power series to arrive at

$$G_i \approx x_i\left(\alpha\psi + \sum_{j=1}^{n}\beta_j x_j\right)$$

for some constants α and β_i. By the symmetry (equivariance) of the system of equations G_i $(1 \le i \le n)$, a permutation $x_i \leftrightarrow x_j$ must lead to a permutation $G_i \leftrightarrow G_j$. This entails $\beta_j = \beta$ $(j = i)$ and $\beta_j = \gamma$ $(j \ne i)$ for some constants β and γ. Then we have

$$G_i \approx x_i\left(\alpha\psi + \beta x_i + \gamma\sum_{j\ne i}^{n} x_j\right) \quad (1 \le i \le n), \tag{B.3}$$

$$\frac{\partial G_i}{\partial x_j} \approx \begin{cases} \alpha\psi + 2\beta x_i + \gamma\sum_{j\ne i}^{n} x_j & (i = j), \\ \gamma x_i & (i \ne j). \end{cases} \tag{B.4}$$

The use of the form $\mathbf{x} = \varepsilon(\mathbf{1}_p, \mathbf{0}_{n-p})$ of a bifurcating branch in (27) in (B.3) leads to

$$G_1 = \cdots = G_p \approx \varepsilon\{\alpha\psi + (\beta + (p-1)\gamma)\varepsilon\},$$
$$G_{p+1} = \cdots = G_n = 0.$$

Consequently, a set of equations $G_i = 0$ $(1 \le i \le n)$ is satisfied by the solution curve $\psi \approx -\frac{\beta+(p-1)\gamma}{\alpha}\varepsilon$ in (28). Substituting $\mathbf{x} = \varepsilon(\mathbf{1}_p, \mathbf{0}_{n-p})$ in (27) into (B.4) and using (28), we obtain

$$\hat{J} = \left\{\frac{\partial G_i}{\partial x_j}\right\} = \varepsilon\begin{pmatrix} A_p(\beta,\gamma) & \gamma E_{pq} \\ O & (\gamma-\beta)I_{n-p}\end{pmatrix}.$$

The eigenvalues of the first diagonal block $\varepsilon A_p(\beta,\gamma)$ give λ_1 (repeated once) and λ_2 (repeated $p-1$ times) and the eigenvalues of the second diagonal block $\varepsilon(\gamma-\beta)I_{n-p}$ give λ_3 (repeated $n-p$ times) in (29).

B.6. *Proof of Proposition 7*

For $p = 2, \ldots, n-1$, eigenvalues $\lambda_2 \approx -(\gamma-\beta)\varepsilon$ and $\lambda_3 \approx (\gamma-\beta)\varepsilon$ in (29) exist and have opposite signs for $\gamma \ne \beta$. Accordingly, the associated branches are unstable for $\gamma \ne \beta$.

From (B.4), the Jacobian matrix for the prebifurcation state has an $(n-1)$-times repeated eigenvalue $\alpha\psi$. Because the prebifurcation state is stable for $\psi > 0$, we have $\alpha < 0$.

For (i), by setting $p = 1$ in (29), we have the stability conditions of $\lambda_1 \approx \beta\varepsilon < 0$ and $\lambda_3 \approx (\gamma-\beta)\varepsilon < 0$, i.e. $\gamma < \beta < 0$ since $\varepsilon > 0$. Then from $\alpha < 0$ and (28), which reduces to $\psi \approx -\frac{\beta}{\alpha}\varepsilon$ for this case, it is apparent that $\psi < 0$.

For (ii), by setting $p = n$ in (29), we have the stability conditions of $\lambda_1 \approx \{\beta + (n-1)\gamma\}\varepsilon < 0$ and $\lambda_2 \approx -(\gamma-\beta)\varepsilon < 0$, i.e. $\beta < \gamma$ and $\beta < -(n-1)\gamma$. Then from (28), which reads $\psi \approx -\frac{\beta+(n-1)\gamma}{\alpha}\varepsilon$ for this case, it is apparent that $\psi < 0$.

In summary, a unique stable branch exists for each of the cases (i) and (ii), whereas there are no stable branches in other cases, designated as (iii). For (iii), we have the remaining parameter space of $\beta > 0$ or $-(n-1)\gamma < \beta < 0$.