Externalities of human capital[*]

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Abstract

Investments in human capital are individual and collective choices with significant external effects. Educated parents and friends speed up our human capital accumulation. Skilled colleagues at work increase our own productivity. Sharing experiences with cultured people is a pleasure *per se*.

Introducing such human capital externalities in a stylized model à la Uzawa (1965), we find that growth can be no longer balanced and the equilibrium globally indeterminate.

Growth goes to one and capital attains a ceiling in a finite lapse of time above the critical value of initial labor supply corresponding to the balanced growth path..

In the case of a constant tax rate, the government is recommended to apply a positive rate to speed up the human capital accumulation during a transition to the capital ceiling.

Keywords: human capital, externalities, growth.

JEL codes: C61, D51, D62, I25, O40.

1 Introduction

The notion of human capital was introduced by Adam Smith in 1776 and updated by Arthur Cecil Pigou in 1928. The modern theory of human capital in terms of education and health dates back to Schultz (1961) and Becker (1964). Ben-Porath (1967) paved the way to the research on educational attainment, on-the-job training and wage growth over the life cycle. Uzawa (1965) pioneered the family of growth theories based on human capital accumulation. The potential of human capital as engine of perpetual growth was fully understood in Rosen's empirical contribution (1976) and Lucas' influential model (1988). After Lucas, the literature developed fast with hundred of contributions on both empirical and theoretical grounds.

Broadly speaking, investments in human capital are not only choices with individual

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consequences, but they also bring large external effects with social consequences. Growing in a family with educated parents, or studying in a class with motivated colleagues, increases the own speed of human capital accumulation. Working where the human capital is denser increases our own productivity because the advice and the example of the others matter. Finally, living with educated and fit people is a pleasure *per se*.

Jacobs (1970) and Acemoglu (2009) stress in general the role of human capital externalities suggesting that the concentration of economic activity in cities is both an effect of these externalities and an engine of economic growth through the exchange of ideas among workers and entrepreneurs. Some papers consider the effects of human capital on human capital accumulation. For instance, concerning education and human capital formation, the reader is referred to De la Croix and Doepke (2003), Tamura (2001), Cervellati and Sunde (2005); while, regarding health and human capital accumulation, to De la Croix and Licandro (1999) and Kalemli-Ozcan et al. (2000). Very few papers focus on the human capital impact as a positive externality on human capital accumulation. Among them, Mookherjee et al. (2010) study an overlapping generations economy with a distribution of household locations. Parents decide whether or not to educate their children, but educational decisions are affected by location. Local complementarity in investment incentives stems from aspirations formation, learning spillovers and local public good. Cavalcanti and Giannitsarou (2017) focus on positive network externalities (local peer) on human capital accumulation. They reconsider the trade-off between growth and inequality embedding networks into an endogenous growth model with overlapping generations.

Human capital can also affect technology and preferences.

Following Uzawa (1965), Lucas (1988) considers the human capital as an ingredient of labor productivity and, then, as an engine of perpetual growth. Moreover, in his influential contribution, Lucas introduces also the human capital as a potential productive externality in the production function.

The literature becomes thinner regarding the impact of human capital on preferences. Although sociologists find some evidence about the role of education in life enjoyment, economists seem to neglect the role of human capital in welfare. Among the sociologists, Ross and Wu (1995) find that better educated agents have a more satisfying job. They are also in a healthier condition and control more their lives. Among the economists, Finkelstein et al. (2013) observe a complementarity between health and consumption demand: the empirical evidence suggests that human capital does increase the marginal utility of consumption. From a theoretical point of view, Bosi et al. (2020) study a market economy with human capital in the utility function and the dynamic consequences in the short and long run. However, they do not consider the human capital as a positive externality but only as an individual choice. Conversely, we take also in account these positive external effects.

We introduce these different kinds of human capital externalities in a stylized model à la Lucas (1988) and study how they interact in the choice of time to work versus time to accumulate human capital, and how they affect the stability properties of equilibria and growth. Since we wish to highlight the role of human capital, we simplify our analysis and disregard physical capital.[1] In addition, we do not consider the possibility of leisure or unemployment: individuals spend their time to work or to study, that is to accumulate human capital. In the word history, a secular stagnation can take place when individuals don't work much and don't study much slowing both the accumulations of physical and human capital (see Eichengreen (2015) among others). We do not consider such slow-down in capital accumulation, even if, in our model, growth can be bounded.

Our model is articulated in two parts.

First, (1) we study the effects of human capital externalities on productivity, human capital accumulation and preferences, then, (2) we introduce an educational fiscal policy to increase the speed of human capital accumulation.

In the first part, we consider a simplified framework where human capital externalities do not affect human capital accumulation.

In this case, the Balanced Growth Path (BGP) exists only for a critical value of the initial labor supply. Below this value, trajectories satisfying the dynamic system violate a necessary transversality condition; above, the equilibrium labor supply converges to one (its maximal amount) implying that human capital converges to a stationary ceiling. Solving the system of differential equations, we provide the trajectories of human capital and labor supply as explicit functions of time. In the particular case of BGP, the growth rate increases in agents' patience, learning easiness, productivity and propensity to human capital.

Conversely, positive capital externalities on capital accumulation rules out the BGP. In this case, we prove that the labor supply reaches one in a finite lapse of time and, from this critical date, the households spend all the time at their disposal to work and, therefore, the human capital stops to grow.

In both the cases, with or without effects of capital externalities on capital accumulation, the equilibrium can be globally indeterminate because the labor supply is a non-predetermined variable. This implies that two economies identical in terms of preferences, technologies (of production and human capital accumulation), and the same initial endowment of capital accumulation, may experience different growth paths.

In the second part, we show that taxation matters during transition. The positive effects of educational public spending externalities and, indirectly, of human capital on capital accumulation, as above, rules out the BGP.

When the initial labor supply is below a critical value, we can not exclude that the decrease in labor supply to zero and the unbounded increase in human capital compensates the loss in working

[1] Concerning the interplay between human and physical capital, the interested readers are referred, among others, to the seminal Lucas (1988) for an infinite-horizon framework or to the more recent Davin et al. (2015) for an OLG model.

time to ensure an optimal consumption level. When the initial labor supply exceed a critical value, labor supply increases to one in a finite lapse of time, agents stop to invest in education and the human capital attains a stationary ceiling. As above the equilibrium can be globally indeterminate.

These results raise a question on the optimality of this fiscal policy. A government constrained to announce, with commitment, a constant tax rate forever, will fix a positive tax rate even in the case of bounded growth (ceiling). Public spending (and, so, taxation) matters during transition from the initial stock of capital to the ceiling, speeding up the accumulation of human capital but it has no effect once attained the ceiling. At best, this fiscal policy implements a second best because of market imperfections (positive human capital externalities) and the announcement of a fix tax rate with commitment.

2 Fundamentals

Human capital is a positive externality with a threefold effect. We assume that the average human capital in the society increases: (1) workers' productivity, (2) the (speed of) human capital accumulation, (3) households' utility. For notational parsimony, we omit the time argument in the following variables.

(1) The average human capital enhances the individual productivity. For simplicity, we consider the labor services as a single input. They are the product of the worker's human capital times her working time: hl. Production y is proportional to this input by a factor $A(\bar{h})$:

$$y = f(hl) = A(\bar{h})hl$$

The productivity of labor services is constant for individuals but it increases in the average human capital from a social perspective.

(2) Formally, the external effect on human capital accumulation takes place as follows:

$$\frac{\dot{h}}{h} = B(\bar{h})(1 - l) \tag{1}$$

where h denotes the individual human capital (education and health in short), $\bar{h}$ the average human capital in the society, l the labor supply. The differential equation (1) means that the growth rate of human capital depends on the time spent in intellectual and physical education, which is equal to the whole time at the disposal, normalized to one, less the working time: $1 - l$, with $l \leq 1$ For simplicity, we assume the growth rate to be proportional to education time by a factor $B(\bar{h})$ which is constant for individuals but increasing in the average human capital from a social perspective.

(3) Finally, living in a rich milieu in terms of human capital, where people are educated and fit can be a pleasant experience:

$$u = u(c, h, \bar{h})$$

with $\partial u/\partial \bar{h} \geq 0$. We notice that the felicity u depends on consumption c, individual human capital (indeed educated and fit people seem to enjoy life more) and, as seen above, the surrounding average human capital $\bar{h}$ seen as a positive externality.

These three effects are assumed to be, plausibly, non-negative externalities.

Assumption 1 *$A, B, u \in C^2$ with $A(\bar{h}) > 0$, $B(\bar{h}) > 0$, $A'(\bar{h}) \geq 0$, $B'(\bar{h}) \geq 0$, $\partial u / \partial \bar{h} \geq 0$ for any $\bar{h} > 0$. u is also strictly increasing and strictly concave in (c, h).*

At this stage, the model remains intendedly general. However three main sub-models are nested in the general one according that we consider one by one the above-mentioned external effects:

(1) externality only on production: $B(\bar{h}) = B$, $\partial u / \partial \bar{h} = 0$;

(2) externality only on capital accumulation: $A(\bar{h}) = A$, $\partial u / \partial \bar{h} = 0$;

(3) externality only on preferences: $A(\bar{h}) = A$, $B(\bar{h}) = B$.

In the following, we consider a market economy: first, a basic model where the accumulation of human capital only depends on the individual choice; second, an economy where the government levies a proportional income tax to finance the public spending in physical and intellectual education.

3 A simple model of market economy

There are two types of agents: the firm and the household. Each firm chooses the labor (services) demand, each consumer-worker the consumption demand and the labor (working time) supply. These agents take prices as given. A general equilibrium mechanism determines these prices.

3.1 Demands and supplies

There are many firms with no market power. For simplicity, we consider a continuum $[0,1]$ of infinitely small firms. The program of firm $j \in [0,1]$ is a simple profit maximization:

$$\max_{x_j} \pi_j(x_j)$$

where the profit is given by $\pi_j(x_j) = f(x_j) - w x_j = A(\bar{h}) x_j - w x_j$ and $x_j \equiv h_j l_j$ denotes the amount of labor services demanded by the price-taker firm j. w is the wage per unit of labor services. $\partial \pi_j / \partial x_j = 0$ implies that, at equilibrium, the wage is equal to the productivity of labor services: $w = A(\bar{h})$, which depends on the average human capital in the society.

The household maximizes an intertemporal utility functional

$$\max \int_0^\infty e^{-\theta t} u(c, h, \bar{h}) \quad (2)$$

subject to the law of capital accumulation, the constraint on labor supply and the budget constraint:

$$\frac{\dot{h}}{h} \leq B(\bar{h})(1 - l) \quad (3)$$

$$l \leq 1 \quad (4)$$

$$c = y = whl$$

We observe that, in our simple model, the product y is entirely consumed and there is no physical capital.

Our formulation of the human capital accumulation is close to Uzawa (1965) and Lucas (1988). Indeed, since $B(\bar{h})$ is given, the investment in human capital $\dot{h}$ is linear in h:[2] $\dot{h} = hB(\bar{h})(1-l)$. This approach allows us to solve the system of differential equations and provide the explicit trajectories for human capital and labor supply in the isoelastic case (see Proposition 9 below).

To solve the household's program, we introduce the Hamiltonian:

$$H(h,l,\lambda,t) = e^{-\theta t}u(whl,h,\bar{h}) + \lambda hB(\bar{h})(1-l) + \nu(1-l) \tag{5}$$

and we apply the Pontryagin's maximum principle.

Proposition 1 (necessary conditions) *Under Assumption 1, the first-order conditions of program (2) are given by*

$$\mu = \frac{w}{B(\bar{h})}\frac{\partial u}{\partial c} - \frac{\xi}{hB(\bar{h})} \tag{6}$$

$$\frac{\dot{\mu}}{\mu} = \theta - B(\bar{h})(1-l) - \frac{1}{\mu}\left(\frac{\partial u}{\partial c}wl + \frac{\partial u}{\partial h}\right) \tag{7}$$

$$\frac{\dot{h}}{h} = B(\bar{h})(1-l) \tag{8}$$

jointly with $\min\{\xi, 1-l\} = 0$ and the transversality condition

$$\lim_{t\to\infty} e^{-\theta t}\mu(t)h(t) \in \mathbb{R}_+ \tag{9}$$

where $\mu \equiv \lambda e^{\theta t}$ is the shadow price of human capital and $\xi \equiv \nu e^{\theta t}$.

Proof. See the Appendix. ■

Equation (6) says that the multiplier is an adjusted marginal utility. Equation (7) captures the intertemporal arbitrage (Euler equation), equation (8) is the law of motion with equality.

Remark 2 (sufficient conditions) *The Mangasarian (1966) condition states that the Pontryagin necessary conditions and the concavity of the Hamiltonian function with respect to the state and control variables, are sufficient for optimality. In our case, the Mangasarian condition does not necessarily hold because the term* $\lambda hB(\bar{h})(1-l)$ *in the Hamiltonian function (5) is not concave in* (h,l). *Arrow (1968) provides a less demanding condition that we will check in the particular case of isoelastic fundamentals and no human capital externalities on human capital accumulation (Proposition 5 below).*

[2] Lucas (1988) also considered, *a priori*, the possibility of $\dot{h}$ being concave on h, but he put aside this parameterization since there would be no endogenous growth at all.

Remark 3 (transversality condition) *The usual transversality condition* $lim_{t\to\infty}\lambda(t)h(t) = 0$ *does not necessarily hold. The transversality condition* $lim_{t\to\infty}\lambda(t)h(t) \in \mathbb{R}$ *is necessary for optimization (see, among others, Bosi et al. (2020), proof of Proposition 1), while* $lim_{t\to\infty}\lambda(t)h(t) = 0$ *is a sufficient condition for an interior solution* (h, l) *to system (6)-(8) satisfying the above concavity conditions (Pontryagin et al., 2018, page 49). Many economic papers select the equilibrium trajectories using the transversality condition* $lim_{t\to\infty}\lambda(t)h(t) = 0$ *without proving that this condition is necessary for optimality. Halkin provides a mathematical example, but meaningless in economic terms (see the footnote at page 46 in Arrow and Kurz (1970) and the counterexample at page 271 in Halkin (1974)). When the objective function and the law of motion are continuously differentiable, a necessary transversality condition is* $lim_{t\to\infty}H(h(t), l(t), \lambda(t), t) = 0$ *where the trajectory* (h, l) *is solution to system (6)-(8) (see Michel (1982) or Acemoglu (2009, equation (7.68) at page 255)). When the value function is finite, this condition implies the transversality condition:*

$$\lim_{t\to\infty}\lambda(t)\dot{h}(t) = 0$$

(see equation (7.71) at page 255 in Acemoglu (2009)).

Summing up, we can say that transversality conditions $\lim_{t\to\infty}\lambda(t)h(t) \in \mathbb{R}$ and $\lim_{t\to\infty}\lambda(t)\dot{h}(t) = 0$ are necessary under mild assumptions such as the continuous differentiability of objective and constraints: they help to select the solutions among the trajectories satisfying the dynamic system such (6)-(8) in our model. Conversely, the usual transversality condition $\lim_{t\to\infty}\lambda(t)h(t) = 0$ is no longer necessary without more demanding restrictions such as some boundedness condition on the fundamentals (Assumption 7.1, point (3) in Acemoglu (2009) or Assumption 4.3 in Kamigashi (2002)).

As we will see, in our case, the restrictive Acemoglu's Assumption 7.1 does not apply, but, fortunately, the less demanding transversality condition

$$\lim_{t\to\infty}\lambda(t)\dot{h}(t) = 0 \tag{10}$$

works to exclude a large set of trajectories and determine the set of equilibrium trajectories.

3.2 Equilibrium

Let us introduce the elasticities of labor service productivity and human capital accumulation with respect to the human capital externality:

$$a(h) \equiv \frac{hA'(h)}{A(h)} \geq 0 \text{ and } b(h) \equiv \frac{hB'(h)}{B(h)} \geq 0 \tag{11}$$

and the second-order elasticities of utility:

$$\varepsilon_1 \equiv \frac{c\partial^2 u/\partial c^2}{\partial u/\partial c}, \varepsilon_2 \equiv \frac{h\partial^2 u/(\partial c\partial h)}{\partial u/\partial c} \text{ and } \varepsilon_3 \equiv \frac{\bar{h}\partial^2 u/(\partial c\partial \bar{h})}{\partial u/\partial c} \tag{12}$$

The households are identical in terms of endowments and preferences. Then, the individual human capitals are identical and equal to the average human capital: $h = \bar{h}$.

Proposition 4 (dynamic general equilibrium) *At equilibrium, when* $l < 1$ *(interior labor supply), dynamics are driven by a two-dimensional dynamic system:*

$$\frac{\dot{h}}{h} = B(h)(1 - l) \tag{13}$$

$$\frac{\dot{l}}{l} = \frac{1}{\varepsilon_1}\Big[\theta - B(h) - (a(h) - b(h) + [1 + a(h)]\varepsilon_1 + \varepsilon_2 + \varepsilon_3)B(h)(1 - l) - \frac{B(h)}{A(h)}\frac{\partial u/\partial h}{\partial u/\partial c}\Big] \tag{14}$$

and the transversality condition: $\lim_{t\to\infty} e^{-\theta t}\mu(t)h(t) \in \mathbb{R}_+$, where $\varepsilon_i = \varepsilon_i(A(h)hl, h, h)$ with $i = 1,2,3$. The initial human capital $h_0 = h(0)$ is given.

Proof. See the Appendix. ■

Remark that this two-dimensional dynamic system summarizes equilibrium in terms of two variables, human capital and labor services. The first one is a predetermined variable, corresponding to a stock variable whose value in each moment was determined by past decisions. In contrast, labour is a non predetermined variable whose value is chosen under the influence of expectations for future wages. Note that future wages influence the return on human capital available in the future, whose accumulation must be decided in anticipation and influence the amount of time available for the household to provide labour services.

3.3 Constant elasticities

In this case, the elasticities are constant: $a(\bar{h}) = a \geq 0$, $b(\bar{h}) = b \in [0,1)$, $\varepsilon_i(c, h, \bar{h}) = \varepsilon_i$. In other words, we assume with some notational abuse:

$$A(\bar{h}) \equiv A\bar{h}^a, B(\bar{h}) \equiv B\bar{h}^b \text{and} u(c, h, \bar{h}) \equiv c^\alpha h^\beta \bar{h}^{1-\alpha-\beta} \tag{15}$$

with $\alpha, \beta > 0$, $\alpha + \beta \leq 1$.

As promised above, now, we show that the first-order condition of the household's program are not only necessary but also sufficient to maximize her utility. For simplicity, we prove that in the simple case $b = 0$ (no human capital externalities on human capital accumulation).

Proposition 5 (Arrow-Mangasarian condition) *In the isoelastic case (15) with* $b = 0$, *if* $l < 1$ *and* $lim_{t\to\infty}\lambda(t)h(t) = 0$, *the first-order conditions (6) to (8) are necessary and sufficient for utility maximization.*

Proof. See the Appendix. ■

Corollary 6 (dynamic general equilibrium) *In the isoelastic case (15), if* $l < 1$, *dynamic system (13)-(14) boils down to*

$$\frac{\dot{h}}{h} = B(h)(1 - l) \tag{16}$$

$$\frac{\dot{l}}{l} = \frac{B(h)}{1-\alpha}\left[1 + a\alpha - b + \left(b + \frac{\beta}{\alpha} - a\alpha\right) l\right] - \frac{\theta}{1-\alpha} \tag{17}$$

jointly with the transversality condition

$$\lim_{t\to\infty} \alpha e^{-\theta t} h(t)^{1-b+a\alpha} l(t)^{\alpha-1} \frac{A^{\alpha}}{B} \in \mathbb{R}_+ \tag{18}$$

Proof. See the Appendix. ■

In the following, we introduce a plausible restriction.

Assumption 2

$$b + \frac{\beta}{\alpha} - a\alpha > 0$$

For instance, this assumption is satisfied if the elasticity of productive externality of human capital is sufficiently small ($a < \beta/\alpha^2$), if the elasticity of externality of capital accumulation is sufficiently large ($b > a\alpha - \beta/\alpha$) or if the share of consumption in total utility is sufficiently small ($\alpha < \sqrt{\beta/a}$).

Let us introduce the critical labor supply such that

$$\frac{B(h_0)}{1-\alpha}\left[1 + a\alpha - b + \left(b + \frac{\beta}{\alpha} - a\alpha\right) l\right] - \frac{\theta}{1-\alpha} = 0$$

that is a function $\bar{l}$ of the initial human capital:

$$\bar{l} \equiv \frac{\frac{\theta}{B(h_0)} + b - 1 - a\alpha}{b + \frac{\beta}{\alpha} - a\alpha}$$

and the following assumption.

Assumption 3

$$1 + a\alpha - b < \frac{\theta}{B(h_0)} < 1 + \frac{\beta}{\alpha} \tag{19}$$

Under Assumption 2, Assumption 3 is equivalent to $0 < \bar{l} < 1$.

Proposition 7 (steady state) *Let Assumptions 1, 2 and 3 hold.*

(1) If $b = 0$, the steady state is given by the Balanced Growth Path (BGP) with

$$l^* = \frac{\frac{\theta}{B} - 1 - a\alpha}{\frac{\beta}{\alpha} - a\alpha} \in (0,1) \tag{20}$$

$$\frac{\dot{h}}{h} = B(1 - l^*) = B\frac{1 + \frac{\beta}{\alpha} - \frac{\theta}{B}}{\frac{\beta}{\alpha} - a\alpha} \equiv \gamma \in (0, B) \tag{21}$$

which satisfies the transversality condition with $\lim_{t\to\infty}\lambda(t)h(t) = 0$.

(2) If $b > 0$, at the steady state, h is constant over time and $l = 1$.

Proof. See the Appendix. ■

If $b = 0$, the human capital around the agent has no impact on her speed of learning ($b = 0$) and the economy experiences the Balanced Growth Path (BGP). In this case, Assumption 3 implies a positive labor supply and a positive education ($0 < l^* < 1$), and a positive (balanced) growth rate ($\gamma > 1$).

As we will see (Proposition 10 below), if $b > 0$, the economy reaches the steady state (which is not a BGP) after a finite lapse of time T. In this case, $h(t) = h(T) \equiv h_T$ and $l(t) = 1$ for any $t \geq T$. Therefore, capital accumulation stops from date T on.

Proposition 8 (balanced growth path) *If $b = 0$, under Assumptions 1, 2 and 3, the impact of the fundamental parameters on the labor supply and the balanced growth rate is given by:*

$$\frac{dl^*}{d\theta} > 0 \text{and} \frac{d\gamma}{d\theta} < 0 \tag{22}$$

$$\frac{dl^*}{dB} < 0 \text{and} \frac{d\gamma}{dB} > 0 \tag{23}$$

$$\frac{dl^*}{da} < 0 \text{and} \frac{d\gamma}{da} > 0 \tag{24}$$

$$\frac{dl^*}{d\beta} < 0 \text{and} \frac{d\gamma}{d\beta} > 0 \tag{25}$$

Moreover,

$$\frac{dl^*}{d\alpha} > 0 \Leftrightarrow \frac{\frac{\beta}{\alpha} + a\alpha}{\frac{\beta}{\alpha} - a\alpha} > \frac{a\alpha}{\frac{\theta}{B} - 1 - a\alpha} \text{and} \frac{d\gamma}{d\alpha} > 0 \Leftrightarrow \frac{\frac{\beta}{\alpha} + a\alpha}{\frac{\beta}{\alpha} - a\alpha} > \frac{B\frac{\beta}{\alpha}}{B\left(1 + \frac{\beta}{\alpha}\right) - \theta} \tag{26}$$

Proof. See the Appendix. ■

The interpretation of the impacts of the fundamentals on the labor supply l^* and the growth rate γ along the BGP (inequalities (22) to (25)) is straightforward.

When agents are less patient, they spend less time in education ($dl^*/d\theta > 0$) and the growth rate of human capital lowers ($d\gamma/d\theta < 0$). Conversely, a higher learning easiness encourages people to invest more in mental and physical education ($dl^*/dB < 0$), and enhances the growth rate at the end ($d\gamma/dB > 0$). Similarly, a higher productivity of human capital induces agents to educate themselves ($dl^*/da < 0$) and accumulate more human capital with a positive effect on growth ($d\gamma/da > 0$). The growing importance of human capital in their preferences compared to consumption also encourages them to invest in education and human capital ($dl^*/d\beta < 0$) with a positive impact on growth ($d\gamma/d\beta > 0$).

The impact of propensity to consumption (α in inequalities (26)) is non-monotonic because working more directly increases current consumption but, lowering the human capital accumulation and, so, labor productivity across time, indirectly decreases future consumption.

Let us focus now on equilibrium transition, set

$$p \equiv \frac{B}{1-\alpha}\left(\frac{\beta}{\alpha} - a\alpha\right) \tag{27}$$

$$q \equiv \frac{\theta - B(1+a\alpha)}{1-\alpha} \tag{28}$$

and notice that, under Assumptions 2 and 3, $p, q > 0$.

Let us consider the two cases: (1) no external effects on human capital accumulation ($b = 0$), (2) external effects ($b > 0$).

Proposition 9 (no externalities on capital accumulation) *Let $b = 0$ and Assumptions 1, 2 and 3 hold. Let $l^* = \bar{l} = q/p$ be the stationary state.*

(1) If $l_0 < l^*$, the trajectory (h, l) satisfying the dynamic system (16)-17) with the initial condition $l(0) = l_0$ violates the necessary transversality condition (10).

(2) If $l_0 = l^*$, then $l(t) = l^*$ forever and the human capital grows at the constant growth rate $\gamma = B(1 - l^*)$. The economy experiences the Balanced Growth Path (BGP):

$$h(t) = h_0 e^{B(1-l^*)t} \tag{29}$$

The BGP satisfies the transversality condition: $\lim_{t\to\infty} \lambda(t)h(t) = 0$.

(3) If $l_0 > l^*$, then $l(t)$ increases and reaches 1 in a finite lapse of time, while $h(t)$ increases from h_0 to h_T during this period, then stops to grow. More explicitly, we have:

$$l(t) = \frac{l_0 l^*}{l_0 + (l^* - l_0)e^{qt}} \text{ if } 0 \leq t < T \tag{30}$$

$$h(t) = h_0 e^{B(1-l^*)t} \left[\frac{l_0 + (l^* - l_0)e^{qt}}{l^*}\right]^{\frac{B}{p}} \text{ if } 0 \leq t < T \tag{31}$$

$$l(t) = 1 \text{ if } t \geq T \tag{32}$$

$$h(t) = h_T = h_0 \left(l_0^{l^*} \left[\frac{l_0(1-l^*)}{l_0 - l^*}\right]^{1-l^*}\right)^{\frac{B}{q}} \text{ if } t \geq T \tag{33}$$

where

$$T = \frac{1}{q}\ln\frac{l_0(1-l^*)}{l_0-l^*} = \frac{1-\alpha}{\theta-B(1+a\alpha)}\ln\frac{l_0(1-l^*)}{l_0-l^*} > 0 \quad (34)$$

is the critical date from which the household stops to accumulate human capital. The transversality condition is satisfied with $\lim_{t\to\infty}\lambda(t)h(t) = 0$.

Proof. See the Appendix. ■

We observe that the usual transversality condition $\lim_{t\to\infty}\lambda(t)h(t) = 0$ is satisfied in cases (2) (BGP) and (3) (bounded growth), while the necessary transversality condition $\lim_{t\to\infty}\lambda(t)\dot{h}(t) = 0$ is violated in case (1) ruling out the corresponding solutions to dynamic system (16)-17) as equilibrium solutions.

As observed before, l is a non-predetermined variable. We are in a market economy where the households observe the prices, but ignore the technology and the initial distribution of human capital and preferences across the population. Thus, they form expectations about the equilibrium and choose l_0 according their beliefs. In this respect, Proposition 9 considers the entire range of initial equilibrium values of l_0 as a potential result of beliefs and market clearing. Thus, l_0 becomes an equilibrium outcome.

From an economic point of view, Proposition 9 shows that economies with a high level of working hours may face eventually a secular stagnation. A high number of working hours (as for instance in Mexico) may be associated with low levels of growth.

Dynamics are different if the surrounding human capital has a positive effect on the speed of individual human capital accumulation ($b > 0$).

Proposition 10 (positive externalities on capital accumulation) *Under Assumptions 2 and 3, if* $b > 0$, *the labor supply* l *will reach* 1 *in a finite lapse of time:* $l(T) = 1$. *From date* T *on, the human capital no longer grows:* $h(t) = h(T) \equiv h_T$ *for any* $t \geq T$, *and the households spend all their time to work:* $l(t) = 1$ *for any* $t \geq T$.

Proof. See the Appendix. ■

Remark 11 (instability of BGP) *According to Proposition 9 and 10, we observe that the BGP requires two demanding conditions in order to exists: no external effects on human capital accumulation (*$b = 0$*) and the coordination of all agents to choose the same critical initial labor supply (*$l_0 = l^*$*). Both these conditions are non-generic and, thus, the BGP is non-generic as well as equilibrium solution. In addition, even in the particular case* $b = 0$, *the BGP is unstable, that is the economy does not converge to the BGP in the long run if* $l_0 \neq l^*$. *Indeed, when* $l_0 > l^*$, *the labor supply reaches one in a finite lapse of time. In the basic* AK *model (Rebelo, 1991), with physical* *capital accumulation, the BGP is the unique solution because the other trajectories violate either the transversality condition or a non-negativity constraint. Rational agents choose the BGP and the BGP is a robust solution. A number of papers on human capital accumulation focus only on the BGP without considering other possible transitional dynamics, and recommend educational policies without addressing the fundamental questions of genericity and robustness of the BGP solution. Public authorities should be concerned by the lack of genericity and robustness of the BGP: in this case, any policy intervention is a potential source of instability. We will deepen this question in Section 4.*

Growth is driven by human capital accumulation and the accumulation mechanism is very sensitive to human capital externalities. We observe that even arbitrarily small externalities ($b > 0$) have a snowball effect in Proposition 10 whatever the initial labor supply, entailing a faster convergence of capital to the ceiling h_T and of labor supply to one in a finite lapse of time: externalities accelerate the increase in human capital, labor productivity (wage) leading workers to supply more. In other words, case (3) of Proposition 9 is amplified by these externalities and becomes the only case of Proposition 10.

h_0 is given, while l_0 is a non-predetermined variable. If $b = 0$, according to Proposition 9, whatever initial condition $l_0 \in [l^*, 1]$ determines a feasible trajectory (h, l) with $l(t) \in [l^*, 1]$ and $h(t) > 0$ and for any $t \geq 0$. Any trajectory, satisfies also the transversality condition $\lim_{t\to\infty} \lambda(t)h(t) = 0$. If $b > 0$, according to Proposition 10, whatever initial condition $l_0 \in [0,1]$ determines a feasible trajectory (h, l) with $l(t) \in [0,1]$ and $h(t) > 0$ and for any $t \geq 0$. Moreover, $h(t) = h_T$ and $l(t) = 1$ for any $t \geq T$, where T is a critical date. Any trajectory, satisfies also the transversality condition $\lim_{t\to\infty} \lambda(t)h(t) \in \mathbb{R}_+$.

Remark 12 (about global indeterminacy) *When agents can choose different initial labor supplies, the initial condition is a distribution of individual choices even if the agents have the same endowments and preferences. Suppose for simplicity that all the agents make the same choice* l_0. *Proposition 9 gives the trajectories satisfying the dynamic system (16)-(17), the initial condition* $h(0) = h_0$ *and the transversality condition* $lim_{t\to\infty} \lambda(t)h(t) = 0$. *Any trajectory is determined by the initial choice* $l(0) = l_0 \in [l^*, 1]$. *Since* l_0 *is non-predetermined, there are multiple trajectories corresponding to all the possible initial choices* l_0. *In the case the agents choose the same* $l_0 \geq l^*$, *this value and the market clearing determine the price path* $w_{l_0} = w_{l_0}(t)$. *If these prices are announced by the auctioneer and agents reply choosing exactly* l_0, *we have a self-consistent solution which is also a self-fulfilling prophecy. In other terms, we can consider* l_0 *as a fixed point of the composition* $l_0 \to w_{l_0} \to l'_0$ *where* $l'_0 = l'_0(w_{l_0})$ *is the consumers' best reply when* $l'_0 = l_0$. *The compactness of* $[l^*, 1] \ni l_0$ *and the continuity properties of the model entail the existence of a fixed point. The existence of multiple fixed points, that is multiple self-consistent equilibria, is also possible.*

3.4 Simulations

Let $l_0 > l_1 \equiv l^*$. Notice that, under $b = 0$, $A(\bar{h}) \equiv A\bar{h}^a$, $B(\bar{h}) \equiv B$, $u(c, h, \bar{h}) \equiv c^\alpha h^\beta \bar{h}^{1-\alpha-\beta}$ with $\alpha + \beta \in (0,1]$, and Assumption 3 becomes

$$1 + a\alpha < \frac{\theta}{B} < 1 + \frac{\beta}{\alpha} \tag{35}$$

which implies Assumption 2.

Consider the following yearly data: $a = 1/3$, $\alpha = 3/6$, $\beta = 1/6$, $h_0 = 1$, $\theta = 1/20$. We fix $B = 2\theta/(2 + a\alpha + \beta/\alpha)$ in order to satisfy both inequalities (35).

Figures 1 and 2 correspond to case (3) in Proposition 9 with $l_0 = 3/4 > l^* = 1/2$: the working time goes to one and growth is bounded.

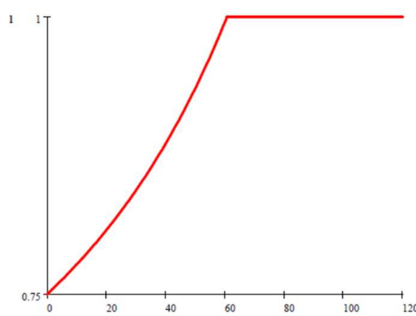

Fig. 1 Working time when $l_0 > l^*$.

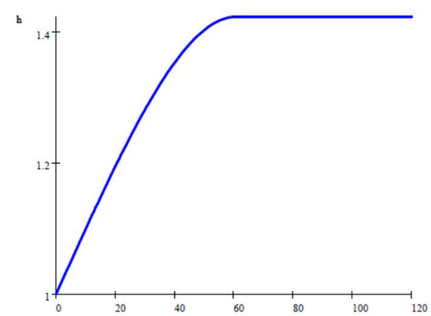

Fig. 2 Human capital when $l_0 > l^*$.

We can plot also the phase diagram. From (62) or (30), we get

$$t = \frac{1}{q} \ln \frac{l_0 l^* - l l_0}{l l^* - l l_0} \tag{36}$$

Replacing (36) in (65) or (31), we obtain the human capital as a function of working time along the equilibrium path:

$$h = \tilde{h}(l) \equiv h_0 \left(\frac{l_0}{l}\right)^{\frac{B}{p}} \left(\frac{q l_0 - p l l_0}{q l - p l l_0}\right)^{B\frac{p-q}{pq}}$$

that is the equation generating the phase diagram. Figure 3 illustrates the different trajectories in the (l, h)-space for different initial conditions l_0. The critical point is $l^* = 1/2$ corresponding to the black vertical line. The blue trajectories on the right of the black line correspond to point (3) in Proposition 9 and show the bounded growth of human capital to the ceiling h_T as l increases to one.

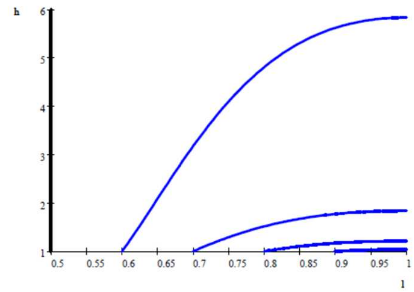

Fig. 3 Phase diagram when $l_0 \geq l^*$ changes

4 Market economy with fiscal policy

We introduce a taxation on (labor) income at the constant rate τ. Taxes allows the government to finance the educational public spending g enhancing the speed of human capital accumulation.

The household still maximizes the utility functional (2), but now her law of motion and budget constraint encompass the fiscal policy:

$$\frac{\dot{h}}{h} = B(g, \bar{h})(1-l) \quad (37)$$

$$l \leq 1$$

$$c = (1-\tau)whl$$

with a positive effect of educational policy on the speed of human capital accumulation: $\partial B / \partial g > 0$. $\bar{h}$ still denotes the average human capital and l the individual labor supply.

The government faces a simple budget constraint: $g = \tau whl$ (taxes are only used to finance physical and intellectual education).

The firms maximizes the profit as above.

4.1 Demands and supplies

The small (price-taker) firm j maximizes as above the profit $\pi_j = f(x_j) - wx_j = A(\bar{h})x_j - wx_j$ with respect to the input of labor services $x_j \equiv h_j l_j$. Profit maximization yields the same first-order conditions: $w = A(\bar{h})$.

To solve the household's program, we apply the Pontryagin's maximum principle. The agent maximizes the intertemporal utility functional (2) under the capital accumulation law (37). The Hamiltonian function now becomes: $H(h, l, \lambda) = e^{-\theta t} u\left((1-\tau)whl, h, \bar{h}\right) + \lambda h B(g, \bar{h})(1-l) + \nu(1-l)$.

Proposition 13 (necessary conditions) *The first-order conditions of the household's program are the following:*

$$\mu = \frac{(1-\tau)w}{B(g,\bar{h})}\frac{\partial u}{\partial c} - \frac{\xi}{hB(g,\bar{h})} \tag{38}$$

$$\frac{\dot{\mu}}{\mu} = \theta - B\left(g,\bar{h}\right)(1-l) - \frac{1}{\mu}\left[\frac{\partial u}{\partial c}(1-\tau)wl + \frac{\partial u}{\partial h}\right] \tag{39}$$

$$\frac{\dot{h}}{h} = B\left(g,\bar{h}\right)(1-l) \tag{40}$$

jointly with $\min\{\xi, 1-l\} = 0$ and the transversality condition $\lim_{t\to\infty} e^{-\theta t}\mu(t)h(t) \in \mathbb{R}_{+}$, where $\mu(t) \equiv \lambda(t)e^{\theta t}$ and $\xi(t) \equiv \nu(t)e^{\theta t}$ are the multipliers.

Proof. See the Appendix. ■

We observe, as above, that the limit condition $\lim_{t\to\infty}\lambda(t)h(t) = 0$ does not necessarily holds.

4.2 Equilibrium

The households are identical in terms of endowments and preferences. If the initial labor supplies are identical, the individual human capitals are also identical and equal to the average human capital: $h = \bar{h}$.

Reconsider the elasticity of labor service productivity (11) depending on h and the second-order elasticities of utility (12) depending on $\left(c, h, \bar{h}\right)$. We introduce the new elasticities of human capital accumulation with respect to the public spending and the human capital externality:

$$b_g\left(g,\bar{h}\right) \equiv \frac{g}{B(g,\bar{h})}\frac{\partial B}{\partial g} \geq 0 \text{and} b_h\left(g,\bar{h}\right) \equiv \frac{\bar{h}}{B(g,\bar{h})}\frac{\partial B}{\partial \bar{h}} \geq 0$$

Proposition 14 (dynamic general equilibrium) *At equilibrium, when* $l < 1$ *(interior labor supply), economic dynamics are driven by a two-dimensional dynamic system:*

$$\frac{\dot{h}}{h} = B(g,h)(1-l) \tag{41}$$

$$\frac{\dot{l}}{l} = \frac{1}{\varepsilon_1 - b_g(g,h)}\left[\theta - B(g,h) - \frac{B(g,h)}{(1-\tau)A(h)}\frac{\partial u/\partial h}{\partial u/\partial c} - \left(a(h) - [1+a(h)]b_g(g,h) - b_h(g,h) + [1+a(h)]\varepsilon_1 + \varepsilon_2 + \varepsilon_3\right)B(g,h)(1-l)\right] \tag{42}$$

and the necessary transversality condition

$$\lim_{t\to\infty} e^{-\theta t}h(t)\frac{(1-\tau)A\left(h(t)\right)}{B\left(\tau A\left(h(t)\right)h(t)l(t), h(t)\right)}\frac{\partial u}{\partial c} \in \mathbb{R}_{+} \tag{43}$$

where $g = \tau A(h)hl$, and $\varepsilon_i = \varepsilon_i\left((1-\tau)A(h)hl, h, h\right)$ with $i = 1,2,3$.

Proof. See the Appendix. ■

We observe that, in our two-dimensional system, the human capital h is a predetermined variable, while the labor supply l is not.

We observe also that, if $\tau = 0$ and $b_g(g,h) = 0$ (no impact of public spending on human capital accumulation), we recover system (13)-(14).

4.3 Constant elasticities

In order to provide an explicit characterization of economic trajectories, we consider constant elasticities: $a(\bar{h}) = a \geq 0$, $b_g(g,\bar{h}) = b_g \geq 0$, $b_h(g,\bar{h}) = b_h \geq 0$ and $\varepsilon_i(c,h,\bar{h}) = \varepsilon_i$. More explicitly, we assume with some notational abuse:

$$A(\bar{h}) \equiv A\bar{h}^a \quad (44)$$

$$B(g,\bar{h}) \equiv Bg^{b_g}\bar{h}^{b_h} \quad (45)$$

and

$$u(c,h,\bar{h}) \equiv c^\alpha h^\beta \bar{h}^{1-\alpha-\beta} \quad (46)$$

with $\alpha,\beta > 0$, $\alpha+\beta \leq 1$.

Proposition 15 (dynamic general equilibrium) *If* $l < 1$, *economic dynamics are driven by the following system*

$$\frac{\dot{h}}{h} = B(g,h)(1-l) \quad (47)$$

$$\frac{\dot{l}}{l} = B(g,h)(r_0 + r_1 l) - \frac{\theta}{1-\alpha+b_g} \quad (48)$$

with

$$r_0 \equiv \frac{1+a\alpha-(1+a)b_g-b_h}{1-\alpha+b_g} \text{ and } r_1 \equiv \frac{(1+a)b_g+b_h-a\alpha+\beta/\alpha}{1-\alpha+b_g} \quad (49)$$

and the necessary transversality condition

$$\lim_{t\to\infty} e^{-\theta t}\frac{\alpha[(1-\tau)A]^\alpha}{B(\tau A)^{b_g}}\left(\frac{h^{r_0}}{l}\right)^{1-\alpha+b_g} \in \mathbb{R}_+ \quad (50)$$

Proof. See the Appendix. ■

We observe that, if $b_g = 0$, we recover system (16)-(17).

In the following, to simplify the dynamic analysis, we introduce a plausible assumption.

Assumption 4 $r_0, r_1 > 0$, that is, more explicitly, $a\alpha - \beta/\alpha < (1+a)b_g + b_h < 1 + a\alpha$.

This assumption, for instance, holds when the external effects of human capital (a, b_g and b_h) are positive but not excessive.

Let us introduce the critical labor supply such that $\dot{l} = 0$, that is, according to (44), (45) and (48),

$$\tilde{h}(l) = \left[\frac{\theta}{B(\tau A l)^{b_g}(1-\alpha+b_g)(r_0+r_1 l)}\right]^{\frac{1}{(1+\alpha)b_g+b_h}}$$

with $\tilde{h}'(l) < 0$. Notice that, under Assumption 4, the denominator into the brackets is positive. Consider the inverse function $\tilde{l}(h) \equiv \tilde{h}^{-1}(h)$ and define the critical labor supply

$$\bar{l} \equiv \tilde{l}(h_0) \tag{51}$$

where h_0 is the initial condition. Notice that $\tilde{l}'(h) < 0$ and that $\dot{l} > 0$ if and only if $h > \tilde{h}(l)$ or, equivalently, $l > \tilde{l}(h)$.

Consider now the steady state of system (47)-(48). If $b_g = b_h = 0$, we recover the case (1) of Proposition 7. If $b_g = 0$ and $b_h > 0$, we recover the case (2) of Proposition 7. Let us focus on the remaining cases $b_g > 0$ and $b_h = 0$ and $b_g > 0$ and $b_h > 0$.

Assumption 5

$$h_0 > \left[\frac{\alpha}{\alpha+\beta}\frac{\theta}{B(\tau A)^{b_g}}\right]^{\frac{1}{(1+\alpha)b_g+b_h}} \equiv H \tag{52}$$

Notice that Assumption 5 is equivalent to $\bar{l} \in (0,1)$.

Notice also that, if $b_g = b_h = 0$, Assumption 5 is equivalent to the RHS inequality of (19) in Assumption 3.

Proposition 16 (steady state) *Let Assumptions 4 and 5 hold. If $b_g > 0$ and $b_h \geq 0$, there is no BGP with $\dot{h}/h > 0$. At the steady state, h is constant over time and $l = 1$.*

Proof. See the Appendix. ■

Putting together Proposition 7 and Proposition 16, we see that a BGP exists only in the case $b_g = b_h = 0$, that is the case in which neither the public spending in education nor the human capital externality in the neighborhood have any effect on the speed of capital accumulation. The only possible steady state corresponds to a full working time ($l = 1$) with no educational activities.

We consider now the possible equilibrium transitions according to the initial choice of labor supply.

If $b_g = b_h = 0$, we recover the results in Proposition 9. If $b_g = 0$ and $b_h > 0$, we recover the results in Proposition 10. Let us focus on the remaining case: $b_g > 0$ and $b_h \geq 0$.

Proposition 17 (positive externalities on capital accumulation) *Let Assumptions 4 and 5 hold and assume $b_g > 0$ and $b_h \geq 0$. Any trajectory has a positive starting point $l_0 > 0$.*

(1) If $0 < l_0 < \bar{l}$, then either (1.1) $\dot{l}(t) < 0$ for any t with $\lim_{t\to\infty} l(t) = 0$ or (1.2) l

reaches 1 in a finite lapse of time.

(2) If $\bar{l} \leq l_0 < 1$, then l reaches 1 in a finite lapse of time.

(3) If $l_0 = 1$, then $l_0 = 1$ forever and h remains at h_0.

In cases (1.2), (2) and (3), any equilibrium trajectory satisfies the transversality condition $\lim_{t\to\infty} \lambda(t)h(t) = 0$.

When the labor supply l reaches 1 in a finite lapse of time, say T, we have that, from date T on, the human capital no longer grows: $h(t) = h(T) \equiv h_T$ for any $t \geq T$, and the household spend all her time to work: $l(t) = 1$ for any $t \geq T$.

Proof. See the Appendix. ■

Remark 18 *Interestingly, we cannot rule out the possibility of an equilibrium solution with a decreasing labor supply (case (1.1)) as was the case in Proposition 9 using the transversality condition (10). We do not know whether (10) applies because we do not have the explicit solution as was the case in Proposition 9. From an economic point of view, because of the positive effect of public spending and, possibly, human capital externality on human capital accumulation (*$b_g > 0$ *and* $b_h \geq 0$*), we cannot exclude that the increase in human capital compensates the decrease in working time with a positive effect on consumption over time and on intertemporal utility at the end.*

Comparing case (1.1) in Proposition 17 with the following cases, we conjecture the existence of an initial threshold $\hat{l} \in (0, \bar{l})$ such that, if $0 < l_0 < \hat{l}$, then $\dot{l}(t) < 0$ for any $t \geq 0$ and $\lim_{t\to\infty} l(t) = 0$ provided that this decreasing trajectory is optimal, while, if $\hat{l} < l_0 < 1$, then l reaches 1 in a finite lapse of time. This means that a sufficiently large initial investment in human capital accumulation $(1 - l_0)$ followed by increasing investment in capital accumulation $(1 - l(t))$ can compensate the progressive reduction in working time $(l(t))$. However, a positive labor supply is always required ($\lim_{t\to\infty} l(t) = 0$ with $l(t) > 0$) in order to have a positive disposable income, that is a positive consumption ($c(t) = (1 - \tau)A(h(t))h(t)l(t) > 0$).

Equation (37) shows how the private effort in education given by $1 - l$ is amplified by the public effort g. However, the occurrence of perpetual growth rests on a low initial level of labor supply, corresponding to a large effort in education, allowing for a rapid initial human capital accumulation.

In Proposition 17, as was the case in Proposition 10, the positive externalities of human capital and, now, of public spending on human capital accumulation ($b_h > 0$ or $b_g > 0$) rule out the BGP. The mechanisms and the considerations outlined in Remark 11 still hold.

Proposition 9 is the main contribution in the positive case, while Proposition 17 in the normative case (see Section 4.4). The modelization of fiscal policy is interesting even when it is difficult to provide explicit trajectories and a policy rule through a welfare maximization. In the positive case, we are able to provide the explicit solutions, while in the normative case, we show instead that the introduction of taxation can affect the critical values for dynamics without changing the fundamental mechanisms. In addition, even in this general case, we find explicit conditions for the convergence of human capital to a stationary value in finite time and this condition depends on taxation. In other words, the main mechanisms seem to be robust to a fiscal extension of the paper.

Remark 19 (about global indeterminacy) *Focus, for simplicity, on a distribution of initial labor supplies* l_0 *identical across population.* h_0 *is given, while* l_0 *is a non-predetermined variable: under Assumptions 4 and 5, if* $b_g > 0$ *and* $b_h \geq 0$*, global indeterminacy of equilibrium is possible. According to Proposition 17, any initial value* $l_0 \in (0,1]$ *generate a self-consistent equilibrium control* l *if the solution* (h, l) *to the dynamic system (47)-(48) satisfies also a sufficient transversality condition and the composite mapping* $l_0 \rightarrow w_{l_0} \rightarrow l_0'$*, where* l_0' *is the agents' best reply to the announced equilibrium path of prices* w_{l_0}*, has some fixed point* $l_0 = l_0'$*. We can not rule out the existence of an equilibrium in case (1.1) of Proposition 17 using a necessary tranversality condition as (10). In cases (1.2), (2) and (3) some initial condition* l_0 *generates an equilibrium because the sufficient transversality condition* $lim_{t\rightarrow\infty}\lambda(t)h(t) = 0$ *holds (indeed, consumption and human capital are bounded). The compactness of the support* $[\bar{l}, 1]$ *and the continuity properties of the fundamentals entail the existence of equilibrium as a fixed point* $l_0 = l_0'$*. Multiple fixed points of the mapping are possible depending on the fundamentals. In the case of multiple equilibrium trajectories, the indeterminacy is global.*

4.4 Second best

Taxation allows the government to partially internalize the positive externalities of human capital at work in a market economy. Partially, because, in our model, the government fix the tax rate once and for all and with a solution, by construction, less efficient than that of a planner.

In cases (1.2), (2) and (3) of Proposition 17, labor supply l reaches 1 in a finite lapse of time and human capital attains a ceiling h_T. Public spending in education increases the speed of human capital accumulation and, so, the convergence to the ceiling.

Moreover, since public spending has a positive effect only on the speed of human capital accumulation through the function $B(g, h)$, the positive effect of taxation vanishes from date T on ($\dot{h} = hB(g, h)(1 - l) = 0$ independently on g because $l = 1$), while the negative effect of taxation on consumption persists ($c = (1 - \tau)whl$). In other terms, public spending is no longer "productive" in terms of capital accumulation from date T on and positive taxation is unjustified from date T on. However, in our model, the rule of the game states that the government announces at the very beginning a fix tax rate forever and commit itself to this rate. Under this policy constraint, it is optimal to tax with a positive tax rate to accelerate the transition from h_0 to h_T even if the public spending plays no longer a role after date T. Commitment is a way to circumvent time inconsistency of public policy.

Alternative rules of the game and optimal policies should be considered in future research, namely: (1) the optimal switching, that is the possibility of fixing two optimal tax rates over time and computing the optimal switching date from one to another, or, more ambitiously, (2) the optimal control, that is, the possibility of smoothing taxation over time according to an efficient control $\tau = \tau^*(t)$ maximizing the government's welfare functional subject to the best reply (Euler equation) of price-taker agents to any potential announce $\tau(t)$.

5 Conclusion

We have considered the effects of human capital externalities on productivity, capital accumulation and preferences in a model with and without public spending in education.

We have provided the explicit solutions in the case without public education and without any direct effect of capital externalities on capital accumulation: the Balanced Growth Path (BGP) exists only for a critical value of the initial labor supply; below, a necessary transversality condition is violated and the corresponding solution to the dynamic system is not an equilibrium; above, the equilibrium labor supply converges to one (the maximal amount) and human capital to a stationary ceiling. Conversely, positive capital externalities on capital accumulation or public spending in education rule out the BGP.

We have provided the equilibrium characterization without public spending in education but positive effects of capital on capital accumulation: the labor supply converges to one in a finite lapse of time and human capital reaches a ceiling (bounded growth).

The same happens with public spending if the initial labor supply exceeds a critical value. Below this value, either the labor supply, as solution of the dynamic system, converges to zero and growth is unbounded, or it converges to one in a finite lapse of time and growth is bounded.

In the case of a constant tax rate and a transition to the capital ceiling, the optimal policy is a positive rate to speed up the human capital accumulation.

The possibility of global indeterminacy in the presence of human capital externalities may help to explain why some countries are able to grow, while others do not. The interaction between externalities and the choice of time devoted to work versus capital accumulation seems to be crucial. However, most important is how the time not spent working is devoted to capital accumulation versus leisure. This is an issue to be explored from a positive point of view in future research.

From a normative point of view, alternative optimal policies should be considered in future research such as the optimal switching from a tax rate to another at an optimal date, or the optimal control of taxation, that is an intertemporal smoothing of the tax rate to maximize the social welfare subject to the best reply (Euler equation) of price-taker agents to any potential fiscal announce.

6 Appendix

Proof of Proposition 1.

Deriving the first-order conditions $\partial H/\partial l = 0$, $\partial H/\partial h = -\dot{\lambda}$ and $\partial H/\partial \lambda = \dot{h}$, and defining $\mu \equiv \lambda e^{\theta t}$, we get

$$\frac{\partial H}{\partial l} = e^{-\theta t}\frac{\partial u}{\partial c}wh - \lambda h B(\bar{h}) - \nu = 0$$

$$\frac{\partial H}{\partial h} = e^{-\theta t}\left(\frac{\partial u}{\partial c}wl + \frac{\partial u}{\partial h}\right) + \lambda B(\bar{h})(1-l) = -\dot{\lambda}$$

$$\frac{\partial H}{\partial \lambda} = hB(\bar{h})(1-l) = \dot{h}$$

jointly with $\min\{\nu, 1-l\} = 0$ and the transversality condition $\lim_{t\to\infty}\lambda(t)h(t) \in \mathbb{R}_+$. Noticing that

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{\mu}}{\mu} - \theta$$

we obtain (6) to (8). ■

Proof of Proposition 4.

When $l < 1$, the Karush-Kuhn-Tucker necessary condition $\min\{\xi, 1-l\} = 0$ implies $\xi = 0$ and

$$\mu = \frac{w}{B(\bar{h})}\frac{\partial u}{\partial c} \tag{53}$$

At equilibrium, $w = A(h)$ and, after the substitution of μ in (7) by (53), system (6)-(8) becomes

$$\mu = \frac{A(h)}{B(h)}\frac{\partial u}{\partial c} \tag{54}$$

$$\frac{\dot{\mu}}{\mu} = \theta - B(h) - \frac{B(h)}{A(h)}\frac{\partial u/\partial h}{\partial u/\partial c} \tag{55}$$

$$\frac{\dot{h}}{h} = B(h)(1-l) \tag{56}$$

where $\partial u/\partial c$ and $\partial u/\partial h$ depend now on $(c, h, \bar{h}) = (A(h)hl, h, h)$. Taking the logarithm of (54), we get

$$\ln\mu = \ln A(h) - \ln B(h) + \ln\frac{\partial u}{\partial c}(A(h)hl, h, h)$$

and, deriving with respect to t, we obtain

$$\frac{\dot{\mu}}{\mu} = (a(h) - b(h) + \varepsilon_1[1 + a(h)] + \varepsilon_2 + \varepsilon_3)\frac{\dot{h}}{h} + \varepsilon_1\frac{\dot{l}}{l} \tag{57}$$

where the elasticities ε_i depend on $(c, h, \bar{h}) = (A(h)hl, h, h)$. Replacing (55) and (56) in (57), we find system (13)-(14). ■

Proof of Proposition 5

Consider the Hamiltonian $H(h, l, \lambda, t) = e^{-\theta t}u(whl, h, \bar{h}) + \lambda hB(\bar{h})(1 - l)$, where the average capital $\bar{h}$ is an externality. In order to prove that the first-order conditions jointly with the transversality condition are not only necessary but also sufficient for utility maximization, let us check the Arrow-Mangasarian condition when the equilibrium trajectory satisfies $l < 1$.

The Arrow-Mangasarian condition is given by Arrow and Kurz (1970). Define $H^*(h, \lambda, t)$ to be the maximum of $H(h, l, \lambda, t)$ with respect to l, given h, λ, and t. The Arrow-Kurz theorem says that, if $H^*(h, \lambda, t)$ is concave in h, for given λ and t, then the necessary (first-order) conditions are also sufficient.

We verify the Arrow-Mangasarian condition in the case $b = 0$ with a Cobb-Douglas felicity: $u(c, h, \bar{h}) \equiv c^\alpha h^\beta \bar{h}^{1-\alpha-\beta}$. The Hamiltonian becomes $H(h, l, \lambda, t) = e^{-\theta t}w^\alpha \bar{h}^{1-\alpha-\beta}h^{\alpha+\beta}l^\alpha + \lambda hB\bar{h}^b(1 - l)$ with $\alpha + \beta < 1$, where $C_1 \equiv (w^\alpha \bar{h}^{1-\alpha-\beta}) > 0$ and $C_2 \equiv B(\bar{h}) > 0$ are taken as given by the household.

If $l < 1$, the first-order condition $\partial H/\partial l = 0$ with $\nu = 0$ implies

$$l = \left(e^{-\theta t}h^{\alpha+\beta-1}w^\alpha \bar{h}^{1-\alpha-\beta}\frac{\alpha}{\lambda B}\right)^{\frac{1}{1-\alpha}}$$

and, replacing in H, we obtain

$$H^*(h, \lambda, t) = Ch^{\frac{\beta}{1-\alpha}} + \lambda Bh$$

where

$$C \equiv (1 - \alpha)\left(e^{-\theta t}w^\alpha \bar{h}^{1-\alpha-\beta}\right)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\lambda B}\right)^{\frac{\alpha}{1-\alpha}} > 0$$

Finally, we obtain that H^* is strictly concave at h:

$$\frac{\partial^2 H^*}{\partial h^2} = C\frac{\beta}{1-\alpha}\frac{\alpha+\beta-1}{1-\alpha}h^{\frac{\beta}{1-\alpha}-2} < 0$$

because $\alpha + \beta < 1$. ■

Proof of Corollary 6.

Replacing $\varepsilon_1 = \alpha - 1$, $\varepsilon_2 = \beta$ and $\varepsilon_3 = 1 - \alpha - \beta$ in (13)-(14) and noticing that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$, we get system (16)-(17). Since

$$\lim_{t\to\infty}\lambda(t)h(t) = \lim_{t\to\infty}e^{-\theta t}\mu(t)h(t) = \lim_{t\to\infty}e^{-\theta t}\frac{A(h(t))}{B(h(t))}\frac{\partial u}{\partial c}h(t)$$

we find the transversality condition (18). ■

Proof of Proposition 7.

Setting $\dot{l} = 0$, we obtain

$$l^* = \frac{b - 1 - a\alpha + \frac{\theta}{B(h)}}{b + \frac{\beta}{\alpha} - a\alpha} \tag{58}$$

In our model, always $\dot{h}/h \geq 0$. A Balanced Growth Path (BGP) requires $\dot{h}/h$ to be constant and positive.

Consider the cases: (1) $b = 0$, (2) $b > 0$.

(1) If $b = 0$, then $B(h) \equiv Bh^b = B$ and the Balanced Growth Path (BGP) is given by (20) and (21).

We require $l^* \in [0,1]$, that is

$$0 \leq \frac{\frac{\theta}{B} - a\alpha - 1}{\frac{\beta}{\alpha} - a\alpha} \leq 1 \tag{59}$$

Under Assumption 2, $\beta/\alpha - a\alpha > 0$ and (59) is equivalent to

$$1 + \alpha a \leq \frac{\theta}{B} \leq 1 + \frac{\beta}{\alpha} \tag{60}$$

(notice that, in particular, we require $B < \theta$). We observe that (60) is also equivalent to $\gamma \in [0, B]$. Assumption 3 with $b = 0$ implies (60). More precisely, under Assumption 2, Assumption 3 is equivalent to $l^* \in (0,1)$ and to $\gamma \in (0, B)$.

The transversality condition (18) introduces a restriction in the parameter space. Along the BGP (with $b = 0$), we get $h = h_0 e^{\gamma t}$ and the transversality condition becomes

$$\lim_{t \to \infty} \lambda(t) h(t) = \lim_{t \to \infty} \alpha e^{-\theta t} h(t)^{1 + a\alpha} l(t)^{\alpha - 1} \frac{A^\alpha}{B}$$

$$= e^{[(1 + a\alpha)\gamma - \theta]t} \alpha h_0^{1 + a\alpha} l^{*\alpha - 1} \frac{A^\alpha}{B} \in \mathbb{R}_+$$

that is $(1 + a\alpha)\gamma - \theta \leq 0$ or, equivalently, $(1 + a\alpha)\gamma \leq \theta$. Under Assumptions 2 and 3, $\gamma < B$ and, therefore, $(1 + a\alpha)\gamma < (1 + a\alpha)B < \theta$. Then, under these assumptions, the transversality condition is satisfied along the BGP.

(2) If $b > 0$, since $B(h) \equiv Bh^b$, $B(h)$ grows forever along the BGP and l^* in (58) can not be constant over time, a contradiction. Hence, if $b > 0$, there is no BGP.

If $l = 1$, then $\dot{h} = 0$ and, because of Assumption 3, l remains equal to 1 forever. Indeed, if l decreases, according to (17), we have for $l = 1 - \varepsilon$ and $\varepsilon > 0$ sufficiently small:

$$\frac{\dot{l}}{l} = \frac{B(h)}{1 - \alpha}\left(1 + \frac{\beta}{\alpha}\right) - \frac{\theta}{1 - \alpha} \geq \frac{B(h_0)}{1 - \alpha}\left(1 + \frac{\beta}{\alpha}\right) - \frac{\theta}{1 - \alpha} > 0$$

Then, h remains constant over time. ■

Proof of Proposition 8.

Differentiating (20) and (21), we obtain the impact of the fundamental parameters on the labor supply and the balanced growth rate:

$$\frac{\theta}{l^*}\frac{dl^*}{d\theta}=\frac{\frac{\theta}{B}}{\frac{\theta}{B}-1-a\alpha}>0 \text{and} \frac{\theta}{\gamma}\frac{d\gamma}{d\theta}=-\frac{\theta}{B\left(1+\frac{\beta}{\alpha}\right)-\theta}<0$$

$$\frac{B}{l^*}\frac{dl^*}{dB}=-\frac{\frac{\theta}{B}}{\frac{\theta}{B}-1-a\alpha}<0 \text{and} \frac{B}{\gamma}\frac{d\gamma}{dB}=\frac{B\left(1+\frac{\beta}{\alpha}\right)}{B\left(1+\frac{\beta}{\alpha}\right)-\theta}>0$$

$$\frac{a}{l^*}\frac{dl^*}{da}=\frac{a\alpha\left(\frac{\theta}{B}-1-\frac{\beta}{\alpha}\right)}{\left(\frac{\beta}{\alpha}-a\alpha\right)\left(\frac{\theta}{B}-1-a\alpha\right)}<0 \text{and} \frac{a}{\gamma}\frac{d\gamma}{da}=\frac{a\alpha}{\frac{\beta}{\alpha}-a\alpha}>0$$

$$\frac{\beta}{l^*}\frac{dl^*}{d\beta}=-\frac{\frac{\beta}{\alpha}}{\frac{\beta}{\alpha}-a\alpha}<0 \text{and} \frac{\beta}{\gamma}\frac{d\gamma}{d\beta}=\frac{B\frac{\beta}{\alpha}\left(\frac{\theta}{B}-1-a\alpha\right)}{\left[B\left(1+\frac{\beta}{\alpha}\right)-\theta\right]\left(\frac{\beta}{\alpha}-a\alpha\right)}>0$$

$$\frac{\alpha}{l^*}\frac{dl^*}{d\alpha}=\frac{\frac{\beta}{\alpha}+a\alpha}{\frac{\beta}{\alpha}-a\alpha}-\frac{a\alpha}{\frac{\theta}{B}-1-a\alpha} \text{and} \frac{\alpha}{\gamma}\frac{d\gamma}{d\alpha}=\frac{\frac{\beta}{\alpha}+a\alpha}{\frac{\beta}{\alpha}-a\alpha}-\frac{B\frac{\beta}{\alpha}}{B\left(1+\frac{\beta}{\alpha}\right)-\theta}$$

Noticing that, under Assumptions 1, 2 and 3, if $b=0$,

$$\frac{\theta}{B}-1-a\alpha>0, B\left(1+\frac{\beta}{\alpha}\right)-\theta>0, \frac{\beta}{\alpha}-a\alpha>0, \frac{\theta}{B}-1-\frac{\beta}{\alpha}<0$$

and we obtain inequalities (22) to (26). ■

Proof of Proposition 9.

If $b=0$, then $B(h)=B$, a constant. Then, if $l<1$, according to (17), labor supply dynamics can be studied independently of the human capital:

$$\dot{l}(t)=pl(t)^2-ql(t) \tag{61}$$

where, under Assumptions 2 and 3, p and q are positive constants given by (27) and (28).

The solution of differential equation (61) is given by

$$l(t)=\frac{l_0 l^*}{l_0+(l^*-l_0)e^{qt}} \tag{62}$$

According to Assumption 3, $l^* \in (0,1)$. We observe the following.

(1) If $l_0<l^*$, since $q>0$, $l(t)$ decreases asymptotically to zero: $\lim_{t\to\infty} l(t)=0$.

(2) If $l_0=l^*$, then $l(t)=l^*$ forever. This is the BGP.

(3) If $l_0>l^*$, then $l(t)$ increases and reaches 1 in a finite lapse of time according to (30) and (32).

More precisely, noticing that (16) with $B(h)=B$ is equivalent to

$$[\ln h(\tau)]'=B[1-l(\tau)] \tag{63}$$

replacing the solution (62) in (63),

$$[\ln h(\tau)]'=B\left[1-\frac{l_0 l^*}{l_0+(l^*-l_0)e^{q\tau}}\right] \tag{64}$$

and integrating both the sides of (64) with respect to τ from 0 to t, we obtain the explicit human capital accumulation

$$h(t) = h_0 e^{B(1-l^*)t}\left[\frac{l_0+(l^*-l_0)e^{qt}}{l^*}\right]^{\frac{B}{p}} \tag{65}$$

(1) If $l_0 < l^*$, $l(t)$ decreases asymptotically to zero ($\lim_{t\to\infty} l(t) = 0$) and $h(t)$ grows forever with an increasing growth rate

$$\frac{\dot{h}(t)}{h(t)} = B[1 - l(t)] = B\,\frac{l_0(1-l^*)+(l^*-l_0)e^{qt}}{l_0+(l^*-l_0)e^{qt}}$$

converging to its upper bound: $\lim_{t\to\infty}\left[\dot{h}(t)/h(t)\right] = B$.

According to (6), the transversality condition (18) with $b = 0$ becomes

$$\lim_{t\to\infty}\lambda(t)h(t) = \lim_{t\to\infty}\alpha e^{-\theta t}h(t)^{1+a\alpha}l(t)^{\alpha-1}\frac{A^\alpha}{B} \in \mathbb{R}_+ \tag{66}$$

Replacing (62) and (65) in (66), we see that

$$\lim_{t\to\infty}\lambda(t)h(t) = \lim_{t\to\infty}\left[\alpha e^{-\theta t}h(t)^{1+a\alpha}l(t)^{\alpha-1}\frac{A^\alpha}{B}\right]$$

$$= \frac{\alpha A^\alpha h_0^{1+a\alpha}}{B l_0^{1-\alpha}}\lim_{t\to\infty}\left(e^{[(1+a\alpha)B(1-l^*)-\theta]t}\left[\frac{l_0+(l^*-l_0)e^{qt}}{l^*}\right]^{1-\alpha+(1+a\alpha)\frac{B}{p}}\right)$$

$$= \frac{\alpha A^\alpha h_0^{1+a\alpha}}{B l_0^{1-\alpha}}\left(\frac{l^*-l_0}{l^*}\right)^{\frac{\alpha+\beta B}{\alpha\ \ p}} \in \mathbb{R}_+$$

thus the transversality condition (9) is satisfied. This condition is only necessary and, thus, we cannot exclude the trajectories with $l_0 < l^*$. However, focusing on the alternative transversality condition (10), we find

$$\lim_{t\to\infty}\lambda(t)\dot{h}(t) = \lim_{t\to\infty}\lambda(t)\big(B[1-l(t)]h(t)\big) = B\lim_{t\to\infty}\lambda(t)h(t)$$

$$= \frac{\alpha A^\alpha h_0^{1+a\alpha}}{l_0^{1-\alpha}}\left(\frac{l^*-l_0}{l^*}\right)^{\frac{\alpha+\beta B}{\alpha\ \ p}} > 0$$

Since $\lim_{t\to\infty}\lambda(t)\dot{h}(t) \neq 0$, the necessary transversality condition is violated and, therefore, any trajectory (h, l) satisfying the dynamic system (16)-17) with $l_0 < l^*$ is not optimal.

(2) If $l_0 = l^*$, then, according to (65), the human capital grows at the constant growth rate $\gamma = B(1 - l^*)$ and the economy experiences the Balanced Growth Path (BGP) (29).

As we have seen above, the BGP satisfies the transversality condition (see subsection on the steady state).

(3) If $l_0 > l^*$, then $l(t)$ increases from l_0 to 1 in a finite lapse of time (see (34)) and $h(t)$ increases according to (65) from h_0 to h_T during this period, then stops to grow. It is possible to compute in this case the peak of human capital $h_T = h(T)$ by replacing (34) in (31) to obtain (33).

We observe that, when $l(t)$ reaches 1, it remains equal to 1 forever. Indeed, if, at $T_1 \geq T$,

$l(t)$ starts to decrease continuously from T_1 to T_2, we have $l(T_2) < l(T_1) = 1$. At $T_3 \equiv (T_1 + T_2)/2$, $l(T_3) < 1$ and equation (17) holds. Under Assumption 3, we obtain

$$\frac{\dot{l}(T_3)}{l(T_3)} = \frac{B(h(T_3))}{1-\alpha}\left(1+\frac{\beta}{\alpha}\right) - \frac{\theta}{1-\alpha} \geq \frac{B(h_0)}{1-\alpha}\left(1+\frac{\beta}{\alpha}\right) - \frac{\theta}{1-\alpha} > 0$$

that is a contradiction. Thus, $l(t)$ can not decrease after T.

The usual transversality condition is satisfied. Indeed, reconsidering (66), we see that

$$\lim_{t\to\infty} \lambda(t)h(t) = \lim_{t\to\infty}\left(\alpha e^{-\theta t} h_T^{1-b+a\alpha} \frac{A^\alpha}{B}\right) = 0$$

■

Proof of Proposition 10.

Focus on equation (17) and consider four sub-cases: (1) $0 \leq l_0 < \bar{l}$, (2) $l_0 = \bar{l}$, (3) $\bar{l} < l_0 < 1$, (4) $l_0 = 1$.

(1) If $0 \leq l_0 < \bar{l}$, then, $\dot{l}(0) < 0$.

Assume that l remains forever under $\bar{l}$: $l < \bar{l}$ for any $t \geq 0$. In this case, since $B(h)$ is non-decreasing,

$$\frac{\dot{h}}{h} = B(h)(1-l) > B(h_0)(1-\bar{l})$$

for any $t \geq 0$ and

$$\frac{\dot{B}}{B} = b\frac{\dot{h}}{h} > bB(h_0)(1-\bar{l})$$

for any $t \geq 0$. Then,

$$B(h) > B(h_0)e^{bB(h_0)(1-\bar{l})t} \tag{67}$$

for any $t \geq 0$. Since $B(h)$ grows at a rate greater than a strictly positive constant and

$$\frac{B(h_0)}{1-\alpha}(1+a\alpha-b) - \frac{\theta}{1-\alpha} < 0$$

there exists a critical date $t_1 > 0$ such that

$$\frac{B(h(t_1))}{1-\alpha}(1+a\alpha-b) - \frac{\theta}{1-\alpha} = 0$$

Let t_2 such that

$$\frac{B(h_0)e^{bB(h_0)(1-\bar{l})t}}{1-\alpha}(1+a\alpha-b) - \frac{\theta}{1-\alpha} = 0$$

that is

$$t_2 = \frac{1}{bB(h_0)(1-\bar{l})}\ln\frac{\theta}{B(h_0)(1+a\alpha-b)}$$

Since $B(h) > B(h_0)e^{bB(h_0)(1-\bar{l})t}$, we have $t_2 > t_1$. Then, for any $t \geq t_2$, we have

$$\frac{\dot{l}}{l}=\frac{B(h)}{1-\alpha}\left[1+a\alpha-b+\left(b+\frac{\beta}{\alpha}-a\alpha\right)l\right]-\frac{\theta}{1-\alpha}$$

$$\geq\frac{B(h)}{1-\alpha}(1+a\alpha-b)-\frac{\theta}{1-\alpha}>\frac{B(h_0)e^{bB(h_0)(1-\bar{l})t}}{1-\alpha}(1+a\alpha-b)-\frac{\theta}{1-\alpha}$$

$$\geq\frac{B(h_0)e^{bB(h_0)(1-\bar{l})t_2}}{1-\alpha}(1+a\alpha-b)-\frac{\theta}{1-\alpha}=0$$

Since, for any $t\geq t_2$, $\dot{l}>0$, then

$$1+a\alpha-b+\left(b+\frac{\beta}{\alpha}-a\alpha\right)l$$

increases. (67) holds for any $t\geq 0$. Then, for any $t\geq t_2$,

$$\frac{B\big(h(t)\big)}{1-\alpha}\left[1+a\alpha-b+\left(b+\frac{\beta}{\alpha}-a\alpha\right)l(t)\right]-\frac{\theta}{1-\alpha}$$

$$>\frac{B(h_0)e^{bB(h_0)(1-\bar{l})t}}{1-\alpha}\left[1+a\alpha-b+\left(b+\frac{\beta}{\alpha}-a\alpha\right)l(t_2)\right]-\frac{\theta}{1-\alpha}$$

Consider a date $t_3>t_2$ such that, for a given constant $\varepsilon>0$, we have

$$\frac{B(h_0)e^{bB(h_0)(1-\bar{l})t_3}}{1-\alpha}\left[1+a\alpha-b+\left(b+\frac{\beta}{\alpha}-a\alpha\right)l(t_2)\right]-\frac{\theta}{1-\alpha}>\varepsilon$$

and, then,

$$\frac{\dot{l}}{l}=\frac{B\big(h(t)\big)}{1-\alpha}\left[1+a\alpha-b+\left(b+\frac{\beta}{\alpha}-a\alpha\right)l(t)\right]-\frac{\theta}{1-\alpha}>\varepsilon \qquad (68)$$

for any $t>t_3$. This is in contradiction with our assumption $l<\bar{l}<1$ for any $t\geq 0$ if $l(t_3)>0$. If $l(t_3)=0$ either there is $t_4>t_3$ such $l(t_4)>0$ and (68) applies leading to a contradiction with $l<\bar{l}$ forever, or $l(t)=0$ for any $t\geq t_3$ leading to $c=whl=0$ and $u\big(c,h,\bar{h}\big)\equiv c^{\alpha}h^{\beta}\bar{h}^{1-\alpha-\beta}=0$ for any $t\geq t_3$ in contradiction with the utility maximization.

Therefore, $l=\bar{l}$ at a given date $t_5>0$ and, since $\dot{h}(t_5)>0$ and, with some notational abuse, $\dot{B}(t_5)=B'\big(h(t_5)\big)\dot{h}(t_5)>0$ and, according to (17), $\dot{l}(t)>0$ for $t>t_5$, that is $l(t)>\bar{l}$ for any $t>t_5$. Hence, there exists a constant $\eta>0$ and $t_6>t_5$ such that, according to (17),

$$\frac{\dot{l}}{l}=\frac{B\big(h(t)\big)}{1-\alpha}\left[1+a\alpha-b+\left(b+\frac{\beta}{\alpha}-a\alpha\right)l(t)\right]-\frac{\theta}{1-\alpha}>\eta$$

for any $t\geq t_6$. Since $l(t_6)>\bar{l}>0$ and $\dot{l}(t)/l(t)>\eta>0$ for any $t\geq t_6$, l will reach 1 in a finite lapse of time. Let T be the critical date such that $l(T)=1$. At this date, h stops to grow: $h(t)=h(T)\equiv h_T$ for any $t\geq T$, and, from this date on, the labor supply is maximal: $l(t)=1$ for any $t\geq T$.

(2) If $l_0=\bar{l}$, then $\dot{l}(0)=0$ and $\dot{h}>0$. Thus, $B(h)$ increases and, according to (17), $\dot{l}>$

0 for $t > 0$. Then, $\dot{l}/l$ increases. l will reach $l = 1$ in a finite lapse of time. Let T be this critical date and $h_T \equiv h(T)$. We obtain $l(t) = 1$ and $h(t) = h_T$ for any $t \geq T$. We observe that the transversality condition is satisfied. Indeed,

$$\lim_{t\to\infty} \lambda(t)h(t) = \lim_{t\to\infty} \alpha e^{-\theta t} h(t)^{1-b+a\alpha} l(t)^{\alpha-1} \frac{A^\alpha}{B} = \lim_{t\to\infty} \alpha e^{-\theta t} h_T^{1-b+a\alpha} \frac{A^\alpha}{B} = 0 \quad (69)$$

(3) If $\bar{l} < l_0 < 1$, then $\dot{h}, \dot{l} > 0$ and, according to (17), since h and l increase, then $\dot{l}/l$ increases as well. Then, as above, l will reach $l = 1$ in a finite lapse of time. Let T be this critical date and $h_T \equiv h(T)$. We obtain $l(t) = 1$ and $h(t) = h_T$ for any $t \geq T$. (69) still holds and the transversality condition is satisfied.

(4) If $l_0 = 1$, then, according to (17), the RHS of (19) implies $\dot{l}(0) > 0$. Therefore, $l(t) = 1$ for any $t \geq 0$ and $h(t) = h_0$ for any $t \geq 0$. ■

Proof of Proposition 13.

Deriving the first-order conditions $\partial H/\partial l = 0$, $\partial H/\partial h = -\dot{\lambda}$ and $\partial H/\partial \lambda = \dot{h}$, and using $\mu \equiv \lambda e^{\theta t}$ and $\xi \equiv \nu e^{\theta t}$, we get

$$\frac{\partial H}{\partial l} = e^{-\theta t} \frac{\partial u}{\partial c} (1-\tau) w h - \lambda h B(g, \bar{h}) - \nu = 0$$

$$\frac{\partial H}{\partial h} = e^{-\theta t} \left(\frac{\partial u}{\partial c} (1-\tau) w l + \frac{\partial u}{\partial h} \right) + \lambda B(g, \bar{h})(1-l) = -\dot{\lambda}$$

$$\frac{\partial H}{\partial \lambda} = h B(g, \bar{h})(1-l) = \dot{h}$$

jointly with the transversality condition $\lim_{t\to\infty} \lambda(t)h(t) \in \mathbb{R}_+$ (see Bosi et al. (2020)). Noticing that $\dot{\lambda}/\lambda = \dot{\mu}/\mu - \theta$, we obtain (38)-(40). ■

Proof of Proposition 14.

When $l < 1$, the Karush-Kuhn-Tucker necessary condition $\min\{\xi, 1-l\} = 0$ implies $\xi = 0$ and

$$\mu = \frac{(1-\tau)w}{B(g,\bar{h})} \frac{\partial u}{\partial c} \quad (70)$$

At equilibrium, $w = A(h)$, $g = \tau A(h) h l$ and $B(g, h) = B(\tau A(h) h l, h)$.

After the substitution of μ in (39) by (70), system (38)-(40) becomes

$$\mu = \frac{(1-\tau)A(h)}{B(\tau A(h) h l, h)} \frac{\partial u}{\partial c} \quad (71)$$

$$\frac{\dot{\mu}}{\mu} = \theta - B(\tau A(h) h l, h) - \frac{B(\tau A(h) h l, h)}{(1-\tau)A(h)} \frac{\partial u/\partial h}{\partial u/\partial c} \quad (72)$$

$$\frac{\dot{h}}{h} = B(\tau A(h)hl, h)(1-l) \tag{73}$$

where $\partial u/\partial c$ and $\partial u/\partial h$ depend now on $(c, h, \bar{h}) = ((1-\tau)A(h)hl, h, h)$. Taking the logarithm of (71), we get

$$\ln\mu = \ln(1-\tau) + \ln A(h) - \ln B(\tau A(h)hl, h) + \ln\frac{\partial u}{\partial c}((1-\tau)A(h)hl, h, h)$$

and, deriving with respect to t, we obtain

$$\frac{\dot{\mu}}{\mu} = \left(a(h) - [1+a(h)]b_g(g,h) - b_h(g,h) + [1+a(h)]\varepsilon_1 + \varepsilon_2 + \varepsilon_3\right)\frac{\dot{h}}{h}$$

$$+\left[\varepsilon_1 - b_g(g,h)\right]\frac{\dot{l}}{l} \tag{74}$$

Replacing (72) and (73) in (74), we obtain the two-dimensional dynamic system (41)-(42).

Finally, substituting $\lambda(t) = e^{-\theta t}\mu(t)$ and (71) in $\lim_{t\to\infty}\lambda(t)h(t) \in \mathbb{R}_+$, we get the transversality condition (43). ■

Proof of Proposition 15

Consider the felicity function (46). In this case, the second-order elasticities of utility become $\varepsilon_1 = \alpha - 1$, $\varepsilon_2 = \beta$ and $\varepsilon_3 = 1 - \alpha - \beta$. We observe that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$. According to equation (42), we obtain (48).

According to (43), (44), (45) and (46), we find

$$\lim_{t\to\infty}\lambda(t)h(t) = \lim_{t\to\infty}e^{-\theta t}\mu(t)h(t) = \lim_{t\to\infty}e^{-\theta t}\frac{(1-\tau)A(h)}{B(g,h)}\frac{\partial u}{\partial c}h$$

that is transversality condition (50). ■

Proof of Proposition 16.

If $b_g > 0$ and $b_h \geq 0$, if $l^* \in (0,1)$ is a steady state, then $B(g,h) \equiv B(\tau A)^{b_g}h^{(1+a)b_g+b_h}l^{*b_g} > 0$ and $\dot{h}/h = B(g,h)(1-l^*) > 0$. Then, $B(g,h)$ increases and, according to (48), $\dot{l} \neq 0$, against the assumption that l^* is a steady state.

If $l^* = 0$, then $\dot{h}/h = B(\tau A)^{b_g}h^{(1+a)b_g+b_h}l^{*b_g} = 0$ and h is constant over time. In this case, $c^* = (1-\tau)Ah^{1+a}l^* = 0$ and the steady state is not optimal, a contradiction.

If $l^* = 1$, then $\dot{h}/h = 0$ and $h = h^*$ is constant over time. $l^* = 1$ remains a steady state only if

$$(r_0 + r_1 l^*)B(\tau A)^{b_g}h^{(1+a)b_g+b_h}l^{*b_g} - \frac{\theta}{1-\alpha+b_g} = \frac{\dot{l}}{l} \geq 0$$

that is if $h^* \geq H$, where H is given by (52). ■

Proof of Proposition 17.

We know that $\dot{l} > 0$ if and only if $l > \tilde{l}(h)$ and that $\bar{l} \equiv \tilde{l}(h_0)$.

First we observe that a trajectory starting from $l_0 = 0$ can not be an equilibrium. Indeed, according to (48), $l(t) = 0$ for any t and $c(t) = 0$ for any t. Positive wages imply that $\int_0^\infty e^{-\theta t} u(c, h, \bar{h}) = 0$ is not a maximum, that is a contradiction.

Focus now on equation (48) and consider three cases: (1) $0 < l_0 < \bar{l}$, (2) $\bar{l} \leq l_0 < 1$, (3) $l_0 = 1$, where $\bar{l}$ is given by (51). Notice that Assumption 5 is equivalent to $\bar{l} \in (0,1)$.

(1) If $0 < l_0 < \bar{l}$, then $\dot{l}(0) < 0$ and $\dot{h}(0) > 0$.

There are two sub-cases: either (1.1) $\dot{l}(t) < 0$ for any $t \geq 0$ or (1.2) there is a minimal $T > 0$ such that $\dot{l}(T) \geq 0$.

In case (1.1), since l decreases forever and is non-negative, the limit is non-negative, say m.

If $\lim_{t\to\infty} l(t) = m > 0$, then $\dot{h}/h > \varepsilon \equiv B(\tau A)^{b_g} h_0^{(1+a)b_g+b_h} m^{b_g} (1 - \bar{l}) > 0$ and h increases forever. Then,

$$B(g, h) = B(\tau A)^{b_g} h^{(1+a)b_g+b_h} l^{b_g} > B(\tau A)^{b_g} (h_0 e^{\varepsilon t})^{(1+a)b_g+b_h} m^{b_g}$$

and

$$\frac{\dot{l}}{l} > (r_0 + r_1 m) B(\tau A)^{b_g} (h_0 e^{\varepsilon t})^{(1+a)b_g+b_h} m^{b_g} - \frac{\theta}{1-\alpha+b_g}$$

From a critical $t = T$, $\dot{l} > 0$, a contradiction. Thus, $\lim_{t\to\infty} l(t) = 0$.

In case (1.2), $\dot{h}(T) > 0$ and then $B(g, h)$ increases at T. According to (48), l increases over time and the growth rate $\dot{l}/l$ as well. Then l reaches 1 in a finite lapse of time.

(2) Let $\bar{l} \leq l_0 < 1$ and consider the following two sub-cases.

(2.1) If $l_0 = \bar{l}$, then $\dot{l}(0) = 0$, but h increases and $B(g, h) = B(\tau A)^{b_g} h^{(1+a)b_g+b_h} l^{b_g}$ as well. So, according to (48), $\dot{l} > 0$ for a sufficiently small $t > 0$. Then, $\dot{l}/l$ increases forever and l reaches 1 in a finite lapse of time.

(2.2) If $\bar{l} < l_0 < 1$, then $\dot{l}(0) > 0$, $B(g, h)$ increases and $\dot{l}/l$ increases forever according to (48) with $B(g, h) \equiv B(\tau A)^{b_g} h^{(1+a)b_g+b_h} l^{b_g}$. Thus, in a finite lapse of time, l reaches 1.

(3) If $\bar{l} < 1 = l_0$, according to (48), $\dot{l}(0)/l(0) > 0$, and, hence, $l(t) = 1$ and $h(t) = h_0$ for any $t \geq 0$.

Since capital and consumption are bounded in cases (1.2), (2) and (3), the transversality condition is satisfied with $\lim_{t\to\infty} \lambda(t) h(t) = 0$.

When $l(t)$ reaches one, say at $t = T$, then $l(t) = 1$ for any $t \geq T$. Assume to the contrary that at $T_1 \geq T$, $l(t)$ starts to decrease continuously from T_1 to T_2, we have $l(T_2) < l(T_1) = 1$. At $T_3 \equiv T_1 + \varepsilon < T_2$ with $\varepsilon > 0$, $l(T_3) < 1$ and equation (48) holds:

$$\frac{\dot{l}(T_3)}{l(T_3)} = B(\tau A)^{b_g} h(T_3)^{(1+a)b_g+b_h} l(T_3)^{b_g} [r_0 + r_1 l(T_3)] - \frac{\theta}{1-\alpha+b_g}$$

For ε sufficiently small, we have $l(T_3) \approx 1$ and, according to definitions (49),

$$\frac{\dot{l}(T_3)}{l(T_3)} \approx \frac{1}{1-\alpha+b_g} \left[\frac{\alpha+\beta}{\alpha} B(\tau A)^{b_g} h(T_3)^{(1+a)b_g+b_h} - \theta \right]$$

$$\geq \frac{1}{1-\alpha+b_g}\left[\frac{\alpha+\beta}{\alpha}B(\tau A)^{b_g}h_0^{(1+a)b_g+b_h}-\theta\right]$$

$$> 0$$

under Assumption 5, that is a contradiction. Thus, $l(t)$ can not decrease after T. ■

7 References

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